

# EQUIVARIANT COHOMOLOGY FOR HAMILTONIAN TORUS ACTIONS ON SYMPLECTIC ORBIFOLDS

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**ABSTRACT.** We study Hamiltonian  $R$ -actions on symplectic orbifolds  $[M/S]$ , where  $R$  and  $S$  are tori. We prove an injectivity theorem and generalize Tolman and Weitsman's proof of the GKM theorem [TW] in this setting. The main example is the symplectic reduction  $X//S$  of a Hamiltonian  $T$ -manifold  $X$  by a subtorus  $S \subset T$ . This includes the class of **symplectic toric orbifolds**. We define the equivariant Chen-Ruan cohomology ring and use the above results to establish a combinatorial method of computing this equivariant Chen-Ruan cohomology in terms of **orbifold fixed point data**.

## 1. INTRODUCTION

There has been a flurry of recent work computing a variety of algebraic invariants for orbifolds. In the present paper, we consider **equivariant** invariants of an orbifold equipped with a group action. Our orbifolds arise as global quotients  $[M/S]$  of a manifold by a torus acting with finite stabilizers, and our actions of a torus  $R$  on  $[M/S]$  arise as extensions of the action of  $S$  on  $M$ . We will discuss ordinary and stringy equivariant invariants, and relate our results to the current literature. We include several explicit examples coming from the symplectic reduction construction in symplectic geometry.

Let  $T \cong S^1 \times \cdots \times S^1$  be a compact torus and  $S \subset T$  a subtorus, with the quotient torus  $R := T/S$ . Let  $M$  be a  $T$ -manifold such that  $[M/S]$  is a compact  $R$ -Hamiltonian orbifold in the sense of [LT]. We note that topological invariants of the orbifold  $[M/S]$  should be  $S$ -equivariant invariants of  $M$ . Hence the  $R$ -equivariant cohomology of the orbifold  $[M/S]$  is defined to be

$$H_R^*([M/S], \mathbb{Z}) := H_T^*(M, \mathbb{Z}).$$

Note that on the level of topological spaces,  $H_T^*(M, \mathbb{Q}) = H_R^*(M/S, \mathbb{Q})$  but over  $\mathbb{Z}$  they are not equal in general. See [H] for this kind of comparison.

The main application comes from symplectic reduction. Let  $(X, \omega)$  be a connected, Hamiltonian  $T$ -manifold with moment map  $\mu_T : X \rightarrow \mathfrak{t}^*$ , where  $\mathfrak{t}$  denotes the Lie algebra of  $T$ . We assume that  $\mu_T$  has a component that is proper and bounded below. For a subtorus  $S \subset T$ , containing the proper component, we have the natural inclusion  $\mathfrak{s} \hookrightarrow \mathfrak{t}$  of Lie algebras, and we let  $\pi_S : \mathfrak{t}^* \rightarrow \mathfrak{s}^*$  denote the dual projection. Then  $X$  is also

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a Hamiltonian  $\mathbf{S}$ -manifold with the induced moment map  $\mu_{\mathbf{S}} = \pi_{\mathbf{S}} \circ \mu_{\mathbf{T}} : X \rightarrow \mathfrak{s}^*$ . For a regular value  $a \in \mathfrak{s}^*$ , we let  $M := \mu_{\mathbf{S}}^{-1}(a)$  be the level set of the  $\mathbf{S}$  moment map. The **symplectic reduction**  $X//\mathbf{S} := [M/\mathbf{S}]$  of  $X$  by the action of  $\mathbf{S}$  at  $a$  is a compact **symplectic orbifold** which is Hamiltonian with respect to the residual torus action of  $\mathbf{R} := \mathbf{T}/\mathbf{S}$ , in the sense of [LT].

Classically, if  $Y$  is a compact Hamiltonian  $\mathbf{T}$ -manifold satisfying the GKM conditions, the GKM theorem [GKM] computes the  $\mathbf{T}$ -equivariant cohomology of  $Y$  in terms of the fixed point data. Our first main result is to generalize the GKM theorem to compute  $H_{\mathbf{R}}^*([M/\mathbf{S}], \mathbb{Z})$  by adopting the proof in [TW] to our setting as follows. Please note that the complete technical hypotheses of the theorems appear in the main body of this paper as noted.

**Theorem A** (Theorems 4.10 and 5.5 below). *Let  $[M/\mathbf{S}]^{\mathbf{R}}$  be the suborbifold consisting the 0-dimensional  $\mathbf{R}$ -orbits in  $[M/\mathbf{S}]$ , and let  $[M_1/\mathbf{S}]$  the **orbifold 1-skeleton**, the suborbifold consisting of the 0- and 1-dimensional  $\mathbf{R}$ -orbits in  $[M/\mathbf{S}]$ . We have the following diagram of  $\mathbf{R}$ -equivariant natural inclusions*

$$\begin{array}{ccc} [M/\mathbf{S}] & \longleftarrow & [M_1/\mathbf{S}] \\ & \nwarrow i & \uparrow j \\ & & [M/\mathbf{S}]^{\mathbf{R}} \end{array} .$$

When we take  $\mathbf{R}$ -equivariant cohomology, the image of the injection

$$i^* : H_{\mathbf{R}}^*([M/\mathbf{S}], \mathbb{Q}) \rightarrow H_{\mathbf{R}}^*([M/\mathbf{S}]^{\mathbf{R}}, \mathbb{Q})$$

is the same as the image of

$$j^* : H_{\mathbf{R}}^*([M_1/\mathbf{S}], \mathbb{Q}) \rightarrow H_{\mathbf{R}}^*([M/\mathbf{S}]^{\mathbf{R}}, \mathbb{Q}).$$

The map  $i^*$  is injective in cohomology with integer coefficients, and the images of  $i^*$  and  $j^*$  coincide in cohomology with integer coefficients, when the stabilizer subgroups are connected and the isotropy weights are primitive.

Applying this theorem to compact symplectic toric orbifolds, we obtain the following.

**Theorem B** (Theorem 6.1 below). *Let  $\mathbf{S}$  be an  $(m - n)$ -dimensional subtorus of the  $m$ -dimensional torus  $\mathbf{T}$  which acts on  $\mathbb{C}^m$  canonically coordinate-wise. Let  $\Delta$  be the moment polytope of the compact toric orbifold obtained as the symplectic reduction  $\mathbb{C}^m//\mathbf{S}$  at a regular value. Then*

$$H_{\mathbf{R}}^*(\mathbb{C}^m//\mathbf{S}, \mathbb{Z}) \cong \text{SR}(\Delta)$$

where  $\text{SR}(\Delta)$  is the Stanley-Reisner ring of the polytope  $\Delta$ . Note that this Stanley-Reisner description depends only on the combinatorial type of  $\Delta$ .

In the second part of this paper, we introduce the  $\mathbf{R}$ -equivariant version of the Chen-Ruan orbifold cohomology ring, denoted  $H_{\text{orb}, \mathbf{R}}([M/\mathbf{S}])$ , for a Hamiltonian  $\mathbf{R}$ -orbifold

$[M/\mathbf{S}]$ . We use our orbifold GKM theorem to compute  $H_{orb,R}([M/\mathbf{S}])$  in terms of fixed point data. Our definitions coincide with those found in the literature; we discuss this further at the end of Section 7.

As a vector space, the **R-equivariant Chen-Ruan orbifold cohomology** is defined to be

$$H_{orb,R}([M/\mathbf{S}]) := \bigoplus_{g \in \mathbf{S}} H_R^*([M^g/\mathbf{S}]).$$

As a graded ring, we must add a shifted grading and define a twisted product given by the usual pull-cup-push formula (see, for example, [FG, GHK, JKK])

$$\eta \odot \xi := e_* (e_1^* \eta \cup e_2^* \xi \cup c_M(g, h)) \quad \text{for } (\eta, \xi) \in H_R^*([M^g/\mathbf{S}]) \times H_R^*([M^h/\mathbf{S}]),$$

where  $e_1, e_2, e$  are the obvious inclusions of  $M^{g,h} := M^g \cap M^h$  to  $M^g$ ,  $M^h$ , and  $M^{gh}$  respectively. Here we adapt the formula from [BCS, EJK] to define the virtual class  $c_M(g, h)$ . Namely, for  $(g, h) \in \mathbf{S} \times \mathbf{S}$ , let  $TM|_{M^{g,h}} = \bigoplus_{\lambda \in \text{Hom}(H, \mathbf{S}^1)} W_\lambda$  be the weight decomposition of the tangent bundle of  $M$  restricted to  $M^{g,h}$  where  $H$  is the subgroup of  $\mathbf{S}$  generated by  $g$  and  $h$ . Then the **obstruction bundle** on  $[M^{g,h}/\mathbf{S}]$  is the  $\mathbf{T}$ -equivariant vector bundle

$$\mathcal{R}_M(g, h) = \bigoplus_{\substack{\lambda \neq 0 \\ a_\lambda(g) + a_\lambda(h) + a_\lambda((gh)^{-1}) = 2}} W_\lambda$$

and the corresponding virtual class  $c_M(g, h)$  is defined as the  $\mathbf{T}$ -equivariant Euler class of  $\mathcal{R}_M(g, h)$

$$c_M(g, h) := \mathbf{e}_\mathbf{T}(\mathcal{R}_M(g, h)) \in H_\mathbf{T}(M^{g,h}).$$

The associativity of this product follows immediately from the excess intersection formula and the projection formula, as in [GHK, JKK]:

**Theorem C** (Theorem 7.1 below). *The ring  $H_{orb,R}([M/\mathbf{S}])$  with the twisted product is associative.*

To apply our GKM theorem to the computation of the equivariant Chen-Ruan cohomology, we introduce a new ring  $(\mathcal{NH}_R(\nu[M/\mathbf{S}]^R), \star)$ . As a vector space,

$$\mathcal{NH}_R(\nu[M/\mathbf{S}]^R) := \bigoplus_{g \in \mathbf{S}} H_\mathbf{T}(\nu(F^g \subset M)),$$

where  $F$  is the  $\mathbf{S}$ -invariant submanifold such that  $[M/\mathbf{S}]^R = [F/\mathbf{S}]$ , and  $\nu(F^g \subset M)$  denotes the normal bundle to  $F^g$  in  $M$ . The product  $\star$  is defined in Section 8 using the isotropy data for the  $\mathbf{T}$ -action on each  $\nu(F^g \subset M)$ . We then prove the following two results.

**Theorem D** (Theorem 8.2 below). *The ring  $(\mathcal{NH}_R(\nu[M/\mathbf{S}]^R), \star)$  is associative.*

**Theorem E** (Theorem 8.4 below). *The natural restriction map*

$$H_{orb,R}([M/S]) \rightarrow \mathcal{NH}_R(v[M/S]^R)$$

*is a homomorphism of graded associative rings.*

Thus when this map is injective over  $\mathbb{Q}$  or  $\mathbb{Z}$ , we may apply our orbifold GKM theorem to compute  $H_{orb,R}([M/S])$  using only local isotropy data at the fixed orbifold points.

In the final section, we compute the  $R$ -equivariant Chen-Ruan cohomology for the toric orbifolds and conclude with two presentations of the ring, first as a quotient ring, and second as a subring of a direct sum of polynomial rings.

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## 2. LOCAL NORMAL FORMS

In this section, we recall basics from [LT] adapted to our setting. Our main focus is to provide a local normal form for a Hamiltonian  $R$ -orbifold  $[M/S]$ . Generically to define the Chen-Ruan invariants, it is best to think of orbifolds as stacks. As we are interested in global quotients, though, we do not need to use this full machinery.

A **locally free action** of a group  $G$  on a space  $X$  is one where the stabilizer subgroup  $G_x$  of a point  $x$  is finite for each  $x \in X$ . An **orbifold**  $[M/S]$  in this paper is defined as the quotient of a manifold  $M$  by a locally free action of a compact torus  $S$ . An orbifold chart for  $[M/S]$  may be given as follows: let  $x \in M$  and let  $S_x$  be the finite stabilizer of  $x$  in  $S$ . The group  $S_x$  acts on  $T_x M$  and acts trivially on  $T_x(S \cdot x)$ . Hence the orbifold chart around  $[x/S_x]$  is given by  $\frac{T_x M}{T_x(S_x)}$  with the induced action of  $S_x$ . A **symplectic orbifold**  $([M/S], \omega)$  is an orbifold  $[M/S]$  together with an  $S$ -invariant 2-form  $\omega \in \Omega^2(M)^S$  such that for each  $x \in M$ ,  $\omega$  induces the  $S_x$ -invariant 2-form  $\bar{\omega}$  on the chart  $\frac{T_x M}{T_x(S_x)}$ , which is closed at the origin and non-degenerate, i.e. the kernel of  $\omega_x : T_x M \rightarrow T_x^* M$  is  $T_x(S_x)$ . A torus  $R$  **acts on a symplectic orbifold**  $([M/S], \omega)$  if there is a short exact sequence  $S \hookrightarrow T \twoheadrightarrow R$  of compact tori, and the group  $T$  acts on  $M$  extending the action of  $S$ , and  $\omega \in \Omega^2(M)^T$  is a  $T$ -invariant 2-form.

The  $R$ -action on  $([M/S], \omega)$  is **Hamiltonian** if there is a  $T$ -invariant map  $\mu : M \rightarrow \mathfrak{r}^*$  where  $\mathfrak{r} := \text{Lie } R$ , satisfying the Hamiltonian condition, i.e. for each  $\xi \in \mathfrak{t}$ , the infinitesimal vector field  $\xi_M$  and  $d\mu^\xi$  are related by

$$\omega(\xi_M, -) = d\mu^\xi$$

where  $\mu^\xi : M \rightarrow \mathbb{R}$  is the component of the moment map given by  $x \mapsto \langle \mu(x), \xi \rangle$ . This is well-defined since  $\mathfrak{r}^* = \text{Ann } \mathfrak{s} \subset \mathfrak{t}^*$ .

For a function  $f : M \rightarrow \mathbb{R}$  and a critical point  $x \in M$  of  $f$ , the **Hessian** of  $f$  is the map  $H(f)_x : T_x M \rightarrow T_x^* M$  defined by  $w_x \mapsto d_x(\mathcal{L}_w f)$ , where  $w$  is any local vector field around  $x$  such that  $w|_x = w_x$ . Note that  $H(f)_x$  does not depend on a local extension of  $w_x$  to a vector field.

A function  $[f] : [M/\mathbf{S}] \rightarrow \mathbb{R}$  on an orbifold  $[M/\mathbf{S}]$  is given by an  $\mathbf{S}$ -invariant function  $f : M \rightarrow \mathbb{R}$ . An orbifold point is given by  $[\mathbf{S} \cdot x/\mathbf{S}] = [x/\mathbf{S}_x]$ . An orbifold point  $[x/\mathbf{S}_x] \in [M/\mathbf{S}]$  is a **critical orbifold point** for  $[f]$  if  $d_x f = 0$ , i.e.  $x$  is a critical point for  $f$ . This is a well-defined notion since  $f$  is  $\mathbf{S}$ -invariant. Indeed if  $d_x f = 0$ , then  $\forall y \in \mathbf{S} \cdot x$ ,  $d_y f = 0$ . Let  $F := \text{Crit}(f) \subset M$  be the set of critical points for  $f$ . Then the  $\mathbf{S}$ -action preserves  $F$  and therefore, if  $F$  is a submanifold of  $M$ , then  $[F/\mathbf{S}]$  is a **critical suborbifold** of  $[M/\mathbf{S}]$ . When extending Bott's version of Morse theory to orbifolds, a key hypothesis will be that the critical set is a suborbifold of  $[M/\mathbf{S}]$ .

Let  $[f] : [M/\mathbf{S}] \rightarrow \mathbb{R}$  be a function and  $[x/\mathbf{S}_x] \in [M/\mathbf{S}]$  a critical point. Then  $H(f)_x(w_x) = d_x(\mathcal{L}_w f) = 0$  for  $w_x \in T_x(\mathbf{S}x)$ , since  $\mathcal{L}_w f = 0$  on a neighborhood of  $x$  by the  $\mathbf{S}$ -invariance of  $f$ . Hence  $H(f)_x$  factors through

$$H(f)_x : T_x M \rightarrow \frac{T_x M}{T_x(\mathbf{S} \cdot x)} \rightarrow T_x^* M.$$

Furthermore, the image of  $H(f)_x$  is contained in  $\text{Ann } T_x(\mathbf{S} \cdot x)$ . Thus we can define the induced **orbifold Hessian** for  $[f]$  as the  $\mathbf{S}_x$ -equivariant map

$$\overline{H(f)_x} : \frac{T_x M}{T_x(\mathbf{S} \cdot x)} \rightarrow \left( \frac{T_x M}{T_x(\mathbf{S} \cdot x)} \right)^* = \text{Ann } T_x(\mathbf{S} \cdot x).$$

A critical suborbifold  $[F/\mathbf{S}]$  is **non-degenerate** if  $\forall x \in F$ ,

$$\ker \overline{H(f)_x} = \frac{T_x F}{T_x(\mathbf{S} \cdot x)}.$$

It is obvious that  $[F/\mathbf{S}]$  is non-degenerate if and only if  $F$  is non-degenerate, since  $F$  is non-degenerate when  $\ker H(f)_x = T_x F$  for all  $x \in F$ .

Since our orbifold is presented as a global quotient by a torus, we may explicitly write down the slice theorem and the local normal form, following [LT]. Let  $[M/\mathbf{S}]$  be a compact, connected Hamiltonian  $\mathbf{R}$ -orbifold where  $\mathbf{S} \hookrightarrow \mathbf{T} \twoheadrightarrow \mathbf{R}$  is the torus extension such that  $\mathbf{T}$  acts on  $M$ . The corresponding maps of Lie algebras are  $\mathfrak{s} \hookrightarrow \mathfrak{t} \twoheadrightarrow \mathfrak{r}$ . We summarize what this implies as follows:

- (1) the class  $\omega \in \Omega^2(M)^{\mathbf{S}}$  is actually  $\mathbf{T}$ -invariant;
- (2) the  $\mathbf{S}$ -action on  $M$  is locally free;
- (3) the map  $\mu : M \rightarrow \mathfrak{t}^*$  is a  $\mathbf{T}$ -invariant map;
- (4) we have the Hamiltonian condition  $\omega(\xi_M, -) = d\mu^\xi$ ,  $\forall \xi \in \mathfrak{t}$ ; and
- (5)  $\forall x \in M$ ,  $\omega_x : T_x M \rightarrow T_x^* M$  induces an isomorphism

$$\overline{\omega}_x : T_x M / T_x(\mathbf{S} \cdot x) \rightarrow (T_x M / T_x(\mathbf{S} \cdot x))^* = \text{Ann } T_x(\mathbf{S} \cdot x).$$

For each  $x \in M$ , let  $T_x$  denote the stabilizer in  $T$  of the point  $x$ , and  $\mathfrak{t}_x := \text{Lie}(T_x)$  its Lie algebra.

*Remark 2.1.* For a form  $\alpha \in \Omega^*(M)^\Gamma$  to induce an  $R$ -equivariant form on the orbifold  $[M/S]$ , it must be  $S$ -basic, i.e.  $i_{X_\xi} \alpha = 0$  for each  $\xi \in \mathfrak{s}$ . If we define  $R$ -equivariant forms on  $[M/S]$  as suggested in [LM], this coincides with the space of  $T$ -equivariant forms on  $M$  that are  $S$ -basic.

We now turn to the linear algebra background needed to describe the local normal form for a Hamiltonian  $R$ -action on the orbifold  $[M/S]$ .

**Lemma 2.2.** *For every point  $x \in M$ , there is a (non-canonical)  $T_x$ -equivariant isomorphism*

$$T_x M \cong \text{Ann}(\mathfrak{s} \oplus \mathfrak{t}_x) \oplus (T_x(Tx))^{\perp_{\omega_x}},$$

where the  $T_x$ -action on  $\text{Ann}(\mathfrak{s} \oplus \mathfrak{t}_x)$  is the coadjoint action (and so is trivial, as  $T_x$  is abelian). Thus we have a  $T_x$ -equivariant isomorphism

$$\frac{T_x M}{T_x(Tx)} \cong \text{Ann}(\mathfrak{s} \oplus \mathfrak{t}_x) \oplus \frac{(T_x(Tx))^{\perp_{\omega_x}}}{T_x(Tx)}.$$

*Proof.* If  $\xi \in \mathfrak{t}_x$ , then the vector field  $\xi_M|_x = 0$ , which implies that  $d_x \mu^\xi = 0$  by property (4) above. Thus for any tangent vector  $v_x$ , we have  $d_x \mu(v_x) \in \text{Ann } \mathfrak{t}_x$  since  $\langle d_x \mu(v_x), \xi \rangle = \langle d_x \mu^\xi, v_x \rangle$  for every  $\xi \in \mathfrak{t}_x$ . Therefore the image of  $T_x M$  under the map  $d_x \mu$  lies in

$$\text{Ann } \mathfrak{s} \cap \text{Ann } \mathfrak{t}_x = \text{Ann } (\mathfrak{s} \oplus \mathfrak{t}_x).$$

The map  $d_x \mu$  is actually a surjection. Using property (4),  $d_x \mu^\xi = 0$  implies  $\xi_M|_x \in T_x(Sx)$ . The dual map is  $(d_x \mu)^* : \mathfrak{t}/(\mathfrak{s} \oplus \mathfrak{t}_x) \rightarrow T_x^* M$ . Let  $\tilde{\mathfrak{t}}_x := \{\xi \in \mathfrak{t} \mid \xi_M|_x \in T_x(Sx)\}$  which is the Lie algebra of  $\tilde{T}_x := \{t \in T \mid tx \in Sx\}$  and then it follows that  $\ker(d_x \mu)^* = \tilde{\mathfrak{t}}_x/(\mathfrak{s} \oplus \mathfrak{t}_x)$ . We can show that  $\tilde{T}_x = S \cdot T_x$  and so  $\tilde{\mathfrak{t}}_x = \mathfrak{s} \oplus \mathfrak{t}_x$ . In fact, if  $st \in S \cdot T_x$  then  $stx \in Sx$  and conversely if  $tx \in Sx$  (i.e.  $tx = sx$  for some  $s \in S$ ), then  $t^{-1}s \in T_x$  and so  $t \in S \cdot T_x$ . Thus  $d_x \mu : T_x M \rightarrow \text{Ann } (\mathfrak{s} \oplus \mathfrak{t}_x)$  is surjective. By definition,  $\ker(d_x \mu) = \{v \in T_x M \mid \omega_x(\xi_M|_x, v) = 0, \forall \xi \in \mathfrak{t}\} = (T_x(Tx))^{\perp_{\omega_x}}$ . By choosing a  $T_x$ -invariant splitting of the surjection  $d_x \mu$ , we have the isomorphism. The second claim follows from  $T_x(Tx) \subset (T_x(Tx))^{\perp_{\omega_x}}$ . Indeed,  $\omega_x(\xi_M|_x, \eta_M|_x) = \langle d_x \mu^\xi, \eta_M|_x \rangle = \eta_M(\mu^\xi)|_x = 0$  for each  $\xi, \eta \in \mathfrak{t}$  where the last equality follows since  $\mu^\xi : M \rightarrow \mathbb{R}$  is constant along  $Tx$  and  $\eta_M|_x \in T_x(Tx)$ .  $\square$   $\square$

The following results are applications of standard results from the theory of Lie group actions. They are discussed further in [LT], and we have adapted them to our setting here.

**Theorem 2.3** (Slice theorem for the  $T$ -action on  $M$ ). *For each point  $x \in M$ , there is a  $T$ -invariant neighborhood  $U$  of the orbit  $T \cdot x$  and a  $T$ -equivariant isomorphism*

$$U \cong T \times_{T_x} \frac{T_x M}{T_x(Tx)},$$

where  $\mathbb{T}_x$  acts on  $\mathbb{T} \times \frac{T_x M}{T_x(\mathbb{T}x)}$  by  $(t, [v_x]) \mapsto (ts^{-1}, [s_* v_x])$ .

As a corollary of Lemma 2.2 and the Slice Theorem 2.3, we obtain

**Corollary 2.4** (Local normal form for the  $\mathbb{T}$ -action on  $M$ ). *For each point  $x \in M$ , there is a  $\mathbb{T}$ -invariant neighborhood  $U$  of  $\mathbb{T} \cdot x$  such that we have a  $\mathbb{T}$ -equivariant isomorphism*

$$\Psi : U \cong \mathbb{T} \times_{\mathbb{T}_x} \left( \text{Ann}(\mathfrak{s} \oplus \mathfrak{t}_x) \oplus \frac{(T_x(\mathbb{T}x))^{\perp_{\omega_x}}}{T_x(\mathbb{T}x)} \right).$$

Both Theorem 2.3 and Corollary 2.4 may be rephrased as an orbifold slice theorem and an orbifold local normal form theorem, as in [LT].

**Theorem 2.5** (Orbifold slice theorem for the  $\mathbb{R}$ -action on  $[M/\mathbb{S}]$ ). *For every point  $[x/\mathbb{S}_x] \in [M/\mathbb{S}]$ , there is an  $\mathbb{R}$ -invariant orbifold neighborhood  $[U/\mathbb{S}]$  of  $[x/\mathbb{S}_x]$  such that we have an  $\mathbb{R}$ -equivariant isomorphism*

$$[U/\mathbb{S}] \cong [(\mathbb{T} \times_{\mathbb{T}_x} W)/\mathbb{S}] = \mathbb{R} \times_{\mathbb{R}_{[x/\mathbb{S}_x]}} [W/\mathbb{S}_x],$$

where  $\mathbb{R}_{[x/\mathbb{S}_x]} := \mathbb{T}_x/\mathbb{S}_x$  and  $W := \frac{T_x M}{T_x(\mathbb{T}x)}$ .

**Corollary 2.6** (Orbifold local normal form for the  $\mathbb{R}$ -action on  $[M/\mathbb{S}]$ ). *For each  $[x/\mathbb{S}_x] \in [M/\mathbb{S}]$ , there is an  $\mathbb{R}$ -invariant orbifold neighborhood  $[U/\mathbb{S}]$  of  $[x/\mathbb{S}_x]$  such that we have an  $\mathbb{R}$ -equivariant isomorphism*

$$[U/\mathbb{S}] \cong \mathbb{R} \times_{\mathbb{R}_{[x/\mathbb{S}_x]}} \left( T_1^*(\mathbb{R}/\mathbb{R}_{[x/\mathbb{S}_x]}) \oplus \left[ \frac{(T_x(\mathbb{T}x))^{\perp_{\omega_x}}}{T_x(\mathbb{T}x)} \middle/ \mathbb{S}_x \right] \right)$$

where  $\mathbb{R}_{[x/\mathbb{S}_x]} = \mathbb{T}_x/\mathbb{S}_x$ .

*Remark 2.7.* We can define and derive everything in this section by letting  $\mathbb{S}$  be a subgroup of  $\mathbb{T}$ , not necessarily a (connected) subtorus.

### 3. ORBIFOLD FIXED POINTS AND THE ORBIFOLD 1-SKELETON

In this section, we use the local normal form of Corollary 2.6 to determine the **orbifold fixed points** and **orbifold 1-skeleton** of  $[M/\mathbb{S}]$ .

**3.1. Orbifold fixed points  $[F/\mathbb{S}]$ .** The set  $[M/\mathbb{S}]^{\mathbb{R}}$  of fixed orbifold points in  $[M/\mathbb{S}]$  with respect to the  $\mathbb{R}$ -action is a symplectic suborbifold, cf. [LT, p. 4206]. If we let  $F := \{x \in M \mid \mathbb{T}x = \mathbb{S}x\}$ , then  $[F/\mathbb{S}] = [M/\mathbb{S}]^{\mathbb{R}}$ . Moreover, since  $\mathbb{S}$  acts on  $M$  locally freely, we have that  $x \in F$  if and only if  $\mathfrak{t}/(\mathfrak{s} \oplus \mathfrak{t}_x) = 0$ . Thus the  $\mathbb{T}$ -equivariant local normal form at a point  $x \in F$  becomes  $U_x \cong \mathbb{T} \times_{\mathbb{T}_x} W$  where  $W := \frac{T_x M}{T_x(\mathbb{T}x)}$ . In this section, we compute  $F$  explicitly by using this normal form.

Let  $\mathbb{T}_{x,1}$  be the connected component of  $\mathbb{T}_x$  containing the identity element. Then  $\mathbb{T}_x = \mathbb{T}_{x,1} \mathbb{S}_x$ . In fact, for all  $t_x \in \mathbb{T}_x$ , there are  $t_{x,1} \in \mathbb{T}_{x,1}$  and  $s \in \mathbb{S}$  such that  $t_x = t_{x,1} s$  since  $\mathbb{T} = \mathbb{T}_{x,1} \mathbb{S}$ . Therefore  $s \in \mathbb{T}_x \cap \mathbb{S} = \mathbb{S}_x$ .

**Lemma 3.1.** *For  $\alpha \in \text{Hom}(\mathbb{T}_x, S^1)$ , define  $\bar{\alpha}$  to be the induced map  $\mathbb{T}_x/\mathbb{S}_x \rightarrow S^1/\alpha(\mathbb{S}_x)$ . Then for  $\alpha, \beta \in \text{Hom}(\mathbb{T}_x, S^1)$ ,*

$$\alpha|_{\mathbb{T}_{x,1}} = \beta|_{\mathbb{T}_{x,1}} \Leftrightarrow \overline{\alpha - \beta} = 0.$$

*Proof.* Since  $\mathbb{T}_x = \mathbb{T}_{x,1} \cdot \mathbb{S}_x$ , every element  $t \in \mathbb{T}_x$  can be written as  $t = t_1 \cdot s$  for some element  $t_1 \in \mathbb{T}_{x,1}$  and  $s \in \mathbb{S}_x$ . Thus, if  $\alpha|_{\mathbb{T}_{x,1}} = \beta|_{\mathbb{T}_{x,1}}$ , we have

$$\alpha(t)(\beta(t))^{-1} = \alpha(s)(\beta(s))^{-1} \in (\alpha - \beta)(\mathbb{S}_x).$$

On the other hand, if  $(\alpha - \beta)(\mathbb{T}_{x,1}) \subset (\alpha - \beta)(\mathbb{S}_x)$ , then  $(\alpha - \beta)(\mathbb{T}_{x,1}) = 1$  since  $\mathbb{T}_{x,1}$  is connected and  $\mathbb{S}_x$  is discrete.  $\square$

Recall that in the local normal form, we may identify  $W := \frac{T_x M}{T_x(\mathbb{T}_x)} = \frac{(T_x(\mathbb{T}_x))^{\perp \omega_x}}{T_x(\mathbb{T}_x)}$ . This vector space is equipped with a  $\mathbb{T}_{x,1}$ -action. Let

$$W = \bigoplus_{\lambda \in \text{Hom}(\mathbb{T}_{x,1}, S^1)} \bar{W}_\lambda$$

be the weight decomposition of the  $\mathbb{T}_{x,1}$ -action on  $W$ .

**Lemma 3.2.** *Under the  $\mathbb{T}$ -equivariant local normal form isomorphism  $\Psi$ , for each point  $x \in F$ , we have*

$$F \cap U_x \cong \mathbb{T} \times_{\mathbb{T}_x} \bar{W}_0.$$

*In particular,  $F$  is a manifold.*

*Proof.* Recall that  $\mathbb{T}$  acts on  $[r, v] \in \mathbb{T} \times_{\mathbb{T}_x} W$  by  $t \cdot [r, v] = [tr, v]$ . Moreover, two equivalence classes  $[r_1, v_1] = [r_2, v_2]$  are equal if and only if there is some  $t_x \in \mathbb{T}_x$  such that  $(r_2 t_x^{-1}, t_{x*} v_2) = (r_1, v_1)$ , or equivalently if and only if  $r_2 r_1^{-1} \in \mathbb{T}_x$  and  $(r_2 r_1^{-1})_* v_2 = v_1$ . Therefore

$$\begin{aligned} [r, v] \text{ is in the image of } F \cap U_x \text{ under } \Psi & \\ \Leftrightarrow \forall t \in \mathbb{T}, \exists s \in \mathbb{S}, \text{ such that } t \cdot [r, v] &= s \cdot [r, v], \text{ by definition of } F \\ \Leftrightarrow \forall t \in \mathbb{T}, \exists s \in \mathbb{S}, \text{ such that } st^{-1} \in \mathbb{T}_x &\text{ and } (st^{-1})_* v = v \\ \Leftrightarrow \forall t_x \in \mathbb{T}_x, \exists s_x \in \mathbb{S}_x, \text{ such that } (t_x s_x)_* v &= v. \end{aligned}$$

For the  $(\Leftrightarrow)$  of the last equivalence, we observe that for a given element  $t \in \mathbb{T}$ , the element  $s$  in  $\mathbb{S}$  such that  $s^{-1}t \in \mathbb{T}_x$  is unique up to  $\mathbb{S}_x = \mathbb{S} \cap \mathbb{T}_x$ . Let

$$W_F := \{v \in W \mid \forall t_x \in \mathbb{T}_x, \exists s_x \in \mathbb{S}_x \text{ such that } (t_x s_x)_* v = v\}.$$

Let  $W = \bigoplus_{\alpha \in \text{Hom}(\mathbb{T}_x, S^1)} W_\alpha$ . Then we can also write as  $W_F = \bigoplus_{\alpha \in \text{Hom}(\mathbb{T}_x, S^1)} W_\alpha$ , where the direct sum runs over  $\alpha \in \text{Hom}(\mathbb{T}_x, S^1)$  such that for each  $t_x \in \mathbb{T}_x$ , there is an  $s_x \in \mathbb{S}_x$  such that  $\alpha(t_x s_x) = 1$ . However this condition is equivalent to  $\alpha(\mathbb{T}_x) = \alpha(\mathbb{S}_x)$ , that is,  $\bar{\alpha} = 0$ . Thus by Lemma 3.1,  $W_F = \bar{W}_0$ , as desired.  $\square$



*Remark 3.3.* We may also write  $F$  as

$$\begin{aligned} F &= \left\{ x \in M \mid \xi_M|_x \in T_x(\mathbf{S}x), \forall \xi \in \mathfrak{t} \right\} \\ &= \left\{ x \in M \mid [\xi_M|_x] = 0 \text{ in } \frac{T_x M}{T_x(\mathbf{S}x)}, \forall \xi \in \mathfrak{t} \right\}. \end{aligned}$$

We will need the following technical lemma to prove Theorem 4.10.

**Lemma 3.4.** *Let  $y \in F \cap U_x$  for  $x \in F$ , then  $\mathbb{T}_y = \bigcap_{v_\alpha \neq 0} \ker \alpha$ , where  $[t, v] \in \mathbb{T} \times_{\mathbb{T}_x} \overline{W}_0$  corresponds to  $y$  under the local normal form  $\Psi$ , and  $v_\alpha$  is the  $\alpha$ -component of  $v$  in the weight decomposition*

$$W = \bigoplus_{\alpha \in \text{Hom}(\mathbb{T}_x, S^1)} W_\alpha.$$

*In particular,  $\mathbb{T}_{x,1} = \mathbb{T}_{x',1}$  for all  $x'$  in the connected component of  $F$  containing  $x$ .*

*Proof.* It suffices to show this in the case when  $v = v_\alpha \neq 0$ . If  $r \in \mathbb{T}_y$ , then  $r \cdot [t, v] = [rt, v] = [t, v]$ . Therefore  $r \in \mathbb{T}_x$  and  $\alpha(r) = 1$ , i.e.  $r \in \ker \alpha$ . If  $r \in \ker \alpha \subset \mathbb{T}_x$ , then  $r \cdot [t, v] = [rt, v] = [t, \alpha(r)v] = [t, v]$ , so  $r \in \mathbb{T}_y$ . This proves the first claim.

Since  $\alpha|_{\mathbb{T}_{x,1}} = 0$ ,  $\mathbb{T}_{x,1}$  must be contained in every  $\mathbb{T}_y$ , where  $y$  is in the neighborhood  $U_x$ . For any  $x, x' \in F$  such that  $\exists y \in U_x \cap U_{x'}$ ,  $\mathbb{T}_{x,1}$  and  $\mathbb{T}_{x',1}$  are the connected component of  $\mathbb{T}_y$  containing 1, so they coincide. This proves the latter claim.  $\square$

**3.2. Orbifold 1-skeleton  $[M_1/\mathbf{S}]$ .** The union of 1-dimensional orbits in  $[M/\mathbf{S}]$  corresponds to the union of  $(\dim \mathbf{S} + 1)$ -dimensional orbits in  $M$ : let  $M_1^\circ$  be the set of all points  $x \in M$  such that  $\mathbb{T}x$  is  $(\dim \mathbf{S} + 1)$ -dimensional, so that  $[M_1^\circ/\mathbf{S}]$  is the union of 1-dimensional orbits. The **orbifold 1-skeleton** is by definition  $[(M_1^\circ \cup F)/\mathbf{S}]$ . In this section, we calculate  $M_1^\circ$  explicitly using the normal form and show that the closure  $M_1$  of  $M_1^\circ$  is  $M_1^\circ \cup F$ . We then demonstrate that the closure  $N$  of a connected component  $N^\circ$  of  $M_1^\circ$  is a manifold and  $[N/\mathbf{S}]$  is a Hamiltonian  $\mathbf{R}$ -orbifold. Since  $x \in M_1^\circ$  if and only if  $\mathfrak{t}/(\mathfrak{s} \oplus \mathfrak{t}_x)$  is one-dimensional, the  $\mathbb{T}$ -equivariant isomorphism  $\Psi$  becomes  $U_x \cong \mathbb{T} \times_{\mathbb{T}_x} (\text{Ann}(\mathfrak{s} \oplus \mathfrak{t}_x) \oplus W)$  where  $W := \frac{(T_x(\mathbb{T}x))^{\perp_{\omega_x}}}{T_x(\mathbb{T}x)}$ . We recall that

$$W = \bigoplus_{\lambda \in \text{Hom}(\mathbb{T}_{x,1}, \mathbb{Z})} \overline{W}_\lambda = \bigoplus_{\alpha \in \text{Hom}(\mathbb{T}_x, \mathbb{Z})} W_\alpha$$

are the weight decompositions with respect to the actions of  $\mathbb{T}_{x,1}$  and  $\mathbb{T}_x$  respectively. Note that the notation  $W_\alpha$  denotes a  $\mathbb{T}_x$  representation, whereas  $\overline{W}_\lambda$  denotes a representation of its identity component  $\mathbb{T}_{x,1}$ .

**Lemma 3.5.** *For  $x \in M_1^\circ$ , the isomorphism  $\Psi$  induces*

$$M_1^\circ \cap U_x \cong \mathbb{T} \times_{\mathbb{T}_x} (\text{Ann}(\mathfrak{s} \oplus \mathfrak{t}_x) \oplus \overline{W}_0).$$

*In particular,  $[M_1^\circ/\mathbf{S}]$  is a Hamiltonian  $\mathbf{R}$ -orbifold.*

*Proof.* If  $v \in W_\alpha$  with  $\alpha|_{T_{x,1}} \neq 0$ , then  $T_x \cdot v$  is at least one dimensional. Therefore  $T \cdot [t, a, v] = T \times_{T_x} (\{a\} \oplus T_x v)$  is at least  $(\dim S + 2)$ -dimensional. If  $v \in W_\alpha$  with  $\alpha|_{T_{x,1}} = 0$ , then  $T \cdot [t, a, v] = T \times_{T_x} (a \oplus v) = T \times_{T_x} (\{a\} \oplus \alpha(T_x) \cdot v)$  is exactly  $(\dim S + 1)$ -dimensional since  $\alpha(T_x)$  is discrete. The second claim follows from the fact that the symplectic structure of  $W$  restricts to the symplectic structure on  $\overline{W}_0$ .  $\square$

**Lemma 3.6.** *Let  $M_1$  to be the closure of  $M_1^\circ$  in  $M$ . Then  $M_1 = M_1^\circ \cup F$ .*

*Proof.* For each  $x \in F$ , every neighborhood of  $x$  in  $U_x$  contains a point in  $M_1^\circ$ , namely  $[t, v]$  where  $v \in W_\alpha$  where  $\overline{\alpha} \neq 0$ . And so the orbit of  $[t, v]$  is  $(\dim S + 1)$ -dimensional. Indeed  $T \cdot [t, v] = T \times_{T_x} \alpha(T_x)v$ . Thus  $F \subset M_1$ . If  $x \in M_1 \setminus (M_1^\circ \cup F)$ , then the  $T$ -orbit of  $x$  is more than  $(\dim S + 1)$ -dimensional. However, if  $y \in M$  has an  $m$ -dimensional  $T$ -orbit, then for every point in  $U_y$ , its  $T$ -orbit is at least  $m$ -dimensional. This leads to a contradiction, and so we must have  $M_1 \subset (M_1^\circ \cup F)$ .  $\square$

*Remark 3.7.* For  $x \in F$ , let  $N$  be the closure of a connected component of  $M_1^\circ$  such that  $x \in N$ , then  $N \cap U_x \cong T \times_{T_x} (\overline{W}_0 \oplus \overline{W}_\lambda)$  for some  $\alpha \neq 0 \in \text{Hom}(T_{x,1}, S^1)$ . This implies that  $[N/S]$  is a Hamiltonian  $R$ -orbifold.

#### 4. THE MORSE-BOTT PROPERTY AND INJECTIVITY

We continue to use the notation from Section 2:  $[M/S]$  is a Hamiltonian  $R$ -orbifold with an  $R$ -invariant moment map  $[\mu] : [M/S] \rightarrow \mathfrak{r}^*$ . Let  $\xi \in \mathfrak{t}$  be a rational element and let  $R_1 \subset R$  be the image of  $T_1 := \overline{\{\exp(t\xi), t \in \mathbb{R}\}} \subset T$  in  $R$ . In this section, we show that  $\mu^\xi : M \rightarrow \mathbb{R}$  is Morse-Bott for every rational  $\xi \in \mathfrak{t}$  such that  $\dim R_1 = n + 1$ . This naturally leads to the injectivity theorem for compact Hamiltonian  $R$ -orbifolds.

**Lemma 4.1.** *For any rational element  $\xi$  in  $\mathfrak{t}$ , the map  $\mu^\xi : M \rightarrow \mathbb{R}$  is a Morse-Bott function.*

*Proof.* In order to show that  $\text{Crit}(\mu^\xi)$  is a submanifold of  $M$ , by Lemma 3.2, it suffices to prove that  $\text{Crit}(\mu^\xi)$  coincides with the submanifold which yields the  $R_1$ -fixed suborbifold in  $[M/S]$ . That is, following Remark 3.3, we must show that

$$Q := \text{Crit}(\mu^\xi) = \{x \in M \mid \xi_M|_x \in T_x(Sx)\}.$$

By definition,  $\text{Crit}(\mu^\xi) = \{x \in M \mid d_x \mu^\xi = 0\}$ . The equation  $d_x \mu^\xi = 0$  implies that  $[\xi_M|_x] = \overline{\omega_x}^{-1}(d_x \mu^\xi) = 0$  in  $\frac{T_x M}{T_x(Sx)}$ . Therefore the claim follows.

Now we turn to non-degeneracy. The function  $\mu^\xi$  is non-degenerate if its Hessian satisfies

$$\ker H(\mu^\xi)_x = T_x Q.$$

The Hessian is evaluated at a tangent vector  $v_x \in T_x Q$  by

$$H(\mu^\xi)_x(v_x) = d_x(\mathcal{L}_v \mu^\xi) = \mathcal{L}_v(d\mu^\xi)|_x = 0,$$

since  $d\mu^\xi|_Q = 0$  and  $v_x \in T_x Q$ . Thus  $\ker H(\mu^\xi)_x \supset T_x Q$ . On the other hand, we may identify

$$\begin{aligned} T_x Q &= \{v_x \in T_x M \mid [\xi_M, v]|_x \in T_x(\mathbb{S}x)\} \\ &= \{v_x \in T_x M \mid [\overline{\xi_M}, \overline{v}]|_x = 0\}, \end{aligned}$$

where  $\overline{\xi_M}$  and  $\overline{v}$  are the local vector fields induced on  $\frac{T_x M}{T_x(\mathbb{S}x)}$ . Since  $\overline{\xi_M} = \overline{\omega}^{-1}(d\mu^\xi)$ , we have

$$\begin{aligned} [\overline{\xi_M}, \overline{v}]|_x &= (\mathcal{L}_{\overline{v}} \overline{\omega}^{-1})|_x(d\mu^\xi) + \overline{\omega}_x^{-1}(\mathcal{L}_{\overline{v}}(d\mu^\xi)|_x) \\ &= \overline{\omega}_x^{-1}(H(\mu^\xi)_x(v_x)). \end{aligned}$$

Therefore if  $v_x \in \ker H(\mu^\xi)_x$ , then  $[\overline{\xi_M}, \overline{v}]|_x = 0$ , i.e.  $\ker H(\mu^\xi)_x \subset T_x Q$ .  $\square$

*Remark 4.2.* In particular,  $[\mu^\xi] : [M/\mathbb{S}] \rightarrow \mathbb{R}$  is an orbifold Morse-Bott function. By this, we mean that  $\text{Crit}([\mu^\xi]) = [M/\mathbb{S}]^{\mathbb{R}^1}$  is an orbifold, and  $[\mu^\xi]$  is non-degenerate.

*Remark 4.3.* A further discussion of the local normal form for symplectic toric orbifolds may be found in [GHH, Section 2]. The ideas developed there motivated this project. In the case when  $M$  is the level set for an  $\mathbb{S}$ -moment map on a Hamiltonian  $\mathbb{T}$ -space  $X$ , our Lemma 4.1 is exactly [GHH, Lemma 2.2]. The remarks in the footnote following the local normal form theorem in [GHH, Section 2] also apply here.

The following two lemmata are well-known.

**Lemma 4.4** (Atiyah-Bott over  $\mathbb{Q}$ , c.f. Lemma 7.1 [TW2]). *Let  $E \rightarrow F$  be a  $\mathbb{T}$ -equivariant complex vector bundle over a connected  $\mathbb{T}$ -space  $F$ . Let  $D$  and  $S$  be the  $\mathbb{T}$ -equivariant disk and sphere bundle corresponding  $E$  with a choice of a  $\mathbb{T}$ -invariant metric. Assume that*

(Q1) *there is a subtorus  $\mathbb{T}_1 \subset \mathbb{T}$  that fixes  $F$  pointwise, and acts non-trivially on the fibers.*

*Then the  $\mathbb{T}$ -equivariant Euler class  $e_{\mathbb{T}}(E, \mathbb{Q})$  is a non-zero divisor. In particular, we have the short exact sequence over  $\mathbb{Q}$ :*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathbb{T}}^i(D, S; \mathbb{Q}) & \longrightarrow & H_{\mathbb{T}}^i(D; \mathbb{Q}) & \longrightarrow & H_{\mathbb{T}}^i(S; \mathbb{Q}) \longrightarrow 0 \\ & & \cong_{\text{Thom}} \downarrow & & \cong_{\text{Homot}} \downarrow & & \\ & & H_{\mathbb{T}}^{i-\lambda}(F; \mathbb{Q}) & \xrightarrow{\cup_{e_{\mathbb{T}}(E)}} & H_{\mathbb{T}}^i(F; \mathbb{Q}) & & \end{array}$$

*The same claim holds over  $\mathbb{Z}$  under a stronger assumption on the action of  $\mathbb{T}_1$ :*

(Z1) *there is a subtorus  $\mathbb{T}_1 \subset \mathbb{T}$  that fixes  $F$  pointwise, and each weight*

$$\lambda \in \text{Hom}(\mathbb{T}_1, S^1) \cong \mathbb{Z}^r$$

*is primitive whenever  $W_\lambda \neq \emptyset$ .*

Here recall that an element  $\lambda$  of a  $\mathbb{Z}$ -module  $R$  is *primitive* if and only if  $\lambda = a\lambda'$  for  $a \in \mathbb{Z}$  and  $\lambda' \in R$  implies  $a = \pm 1$ .

**Example 4.5.** If axiom (Z1) is not satisfied, then the Euler class  $e_{\mathbb{T}}(E, \mathbb{Z})$  may be a torsion class in equivariant cohomology with integer coefficients. Let  $F$  be the Klein bottle, or any other space with 2-torsion in its integral cohomology ring. Consider the trivial bundle  $E = \mathbb{C} \times F \rightarrow F$ , equipped with the circle action  $t \cdot (z, f) = (t^2 \cdot z, f)$ , for  $z \in \mathbb{C}$  and  $f \in F$ , which fixes the base pointwise and spins each fiber at double speed. Then  $H_{S^1}^*(F; \mathbb{Z}) = H^*(F; \mathbb{Z}) \otimes \mathbb{Z}[x]$  and under this identification we have  $e_{\mathbb{T}}(E, \mathbb{Z}) = 1 \otimes 2x$ . Let  $\alpha \in H^*(F; \mathbb{Z})$  be a 2-torsion class (i.e.  $2\alpha = 0$ ), then  $\alpha \otimes 1$  is an equivariant class, and

$$\begin{aligned} \alpha \otimes 1 \cup e_{\mathbb{T}}(E, \mathbb{Z}) &= \alpha \otimes 1 \cup 1 \otimes 2x \\ &= \alpha \otimes 2x \\ &= 2\alpha \otimes x \\ &= 0. \end{aligned}$$

Thus,  $e_{\mathbb{T}}(E, \mathbb{Z})$  is a zero divisor.

**Lemma 4.6** (c.f. Lemma 4.4. [LT]). *Let  $f : M \rightarrow \mathbb{R}$  be a  $\mathbb{T}$ -invariant Morse-Bott function on a compact manifold  $M$  equipped with an action of  $\mathbb{T}$ . Choose  $[a, b] \subset \mathbb{R}$  which contains a unique critical value  $c$ . Let  $F$  be the critical submanifold such that  $f(F) = c$ . Let  $E^-$  be its  $\mathbb{T}$ -equivariant negative normal bundle and  $D$  and  $S$  be the corresponding  $\mathbb{T}$ -equivariant disk and sphere bundles. Then  $(M_b^-, M_a^-)$  can be retracted onto the pair  $(D, S)$  so that we have*

$$\cong_{MB} H_{\mathbb{T}}^*(M_b^-, M_a^-; \mathbb{Z}) \cong H_{\mathbb{T}}^*(D, S; \mathbb{Z})$$

where  $M_a^- := f^{-1}(-\infty, a)$ .

By Lemma 4.1, the following generalization of [TW, Proposition 2.1] obviously follows from Lemmata 4.4 and 4.6.

**Proposition 4.7.** *Let  $c \in \mathbb{R}$  be a critical value for  $\mu^{\xi}$  and let  $F_c$  be the set of critical points contained in  $(\mu^{\xi})^{-1}(c)$ . Assume that*

(Q2) *for each connected component  $F'_c$  of  $F_c$ , there is a subtorus of  $\mathbb{T}$ , that fixes  $F'_c$  pointwise and acts non-trivially on the negative normal bundle  $E_c^-|_{F'_c}$ .*

Let  $\epsilon \geq 0$  such that  $c$  is the only critical point in  $[a, b] := [c - \epsilon, c + \epsilon]$ . Then we have the short exact sequence over  $\mathbb{Q}$ :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_{\mathbb{T}}^i(M_b^-, M_a^-; \mathbb{Q}) & \longrightarrow & H_{\mathbb{T}}^i(M_b^-; \mathbb{Q}) & \longrightarrow & H_{\mathbb{T}}^i(M_a^-; \mathbb{Q}) \longrightarrow 0 \\
 & & \cong_{MB} \downarrow & & \downarrow & & \\
 & & H_{\mathbb{T}}^i(D_c, S_c; \mathbb{Q}) & \longrightarrow & H_{\mathbb{T}}^i(D_c; \mathbb{Q}) & & \\
 & & \cong_{Thom} \downarrow & & \cong_{Homot} \downarrow & & \\
 & & H_{\mathbb{T}}^{i-\lambda}(F_c; \mathbb{Q}) & \xrightarrow{\cup_{\mathfrak{e}_{\mathbb{T}}(E_c^-)}} & H_{\mathbb{T}}^i(F_c; \mathbb{Q}) & & 
 \end{array}$$

where  $E_c^-$  is the negative normal bundle for  $F_c$  and  $D_c$  and  $S_c$  are the corresponding  $\mathbb{T}$ -equivariant disk and sphere bundles.

The claim also holds over  $\mathbb{Z}$  if we assume

(Z2) for each connected component  $F'_c$  of  $F_c$ , there is a subtorus of  $\mathbb{T}$  that fixes  $F'_c$  pointwise and acts on the negative normal bundle  $E_c^-|_{F'_c}$  in such a way that each weight  $\lambda \in \text{Hom}(\mathbb{T}_1, S^1) \cong \mathbb{Z}^r$  is primitive ( $r \geq 1$ ).

*Remark 4.8.* Note that the hypothesis (Q2) implies (Q1), and (Z2) implies (Z1).

*Remark 4.9.* We can always choose a rational element  $\xi$  in  $\mathfrak{t}$  such that

$$\{x \in M \mid \xi_M|_x \in T_x(\mathbf{S}x)\} = \{x \in M \mid \xi'_M|_x \in T_x(\mathbf{S}x), \forall \xi' \in \mathfrak{t}\},$$

or equivalently,  $[M/\mathbf{S}]^{\mathbb{R}_1} = [M/\mathbf{S}]^{\mathbb{R}}$ .

**Theorem 4.10.** Let  $F := \{x \in M \mid \mathbb{T}x = \mathbf{S}x\}$  so that  $[M/\mathbf{S}]^{\mathbb{R}} = [F/\mathbf{S}]$ , and let  $i : F \hookrightarrow M$  be the inclusion map. Then the induced map  $i^* : H_{\mathbb{T}}^*(M, \mathbb{Q}) \hookrightarrow H_{\mathbb{T}}^*(F, \mathbb{Q})$  is injective. Equivalently, the induced map  $i^* : H_{\mathbb{R}}^*([M/\mathbf{S}], \mathbb{Q}) \hookrightarrow H_{\mathbb{R}}^*([F/\mathbf{S}], \mathbb{Q})$  is injective.

Moreover, if

(Z3) for each connected component  $F'$  of  $F$ , each weight of the action of  $\mathbb{T}_{F'}$  on the normal bundle  $\nu(F' \subset M)$  is primitive

then  $i^* : H_{\mathbb{T}}^*(M, \mathbb{Z}) \hookrightarrow H_{\mathbb{T}}^*(F, \mathbb{Z})$  is also injective

Although the proof is analogous to the proof given in [TW], we include the complete proof of injectivity here so that we may observe that the proof works in this orbifold set-up, and under certain conditions, it holds with  $\mathbb{Z}$ -coefficients.

*Proof.* For each connected component  $F'$  of  $F$ , the group  $\mathbb{T}_{F'} := \mathbb{T}_{x,1}$ , for some  $x \in F'$ , is the maximal global isotropy subtorus of  $\mathbb{T}$  for  $F'$  by Lemma 3.4. Each weight of the action of  $\mathbb{T}_{F'}$  on the negative normal bundle  $E^-|_{F'}$  is non-trivial, which implies the condition (Q2) in Proposition 4.7. Therefore all lemmata and proposition in this section hold for our setup.

Let  $\xi$  is a generic element in  $\mathfrak{t}$  and  $c_1 < \dots < c_n$  the critical values for  $\mu^\xi$ . Choose a small  $\epsilon > 0$  such that  $c_i$  is the only critical value in  $[a_i, b_i] := [c_i - \epsilon, c_i + \epsilon]$ ,  $\forall i$ .

Let  $M_{a_i}^- := (\mu^\xi)^{-1}(-\infty, a_i)$ ,  $M_{b_i}^- := (\mu^\xi)^{-1}(-\infty, b_i)$ , and  $F_{c_i} := F \cap (\mu^\xi)^{-1}(c_i)$ . We will prove that the restriction map  $H_{\mathbb{T}}^*(M_{b_i}^-) \rightarrow H_{\mathbb{T}}^*(F \cap M_{b_i}^-)$  is injective for all  $i$  and then the theorem follows from the case  $i = n$ , i.e.  $M_{b_n}^- = M$ . We have the following commutative diagram with the exact horizontal rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{\mathbb{T}}^*(M_{b_i}^-, M_{a_i}^-) & \longrightarrow & H_{\mathbb{T}}^*(M_{b_i}^-) & \longrightarrow & H_{\mathbb{T}}^*(M_{a_i}^-) \longrightarrow 0 \\
& & \downarrow \gamma_i & & \downarrow & & \downarrow \beta_i \\
0 & \longrightarrow & H_{\mathbb{T}}^*(F \cap M_{b_i}^-, F \cap M_{a_i}^-) & \longrightarrow & H_{\mathbb{T}}^*(F \cap M_{b_i}^-) & \longrightarrow & H_{\mathbb{T}}^*(F \cap M_{a_i}^-) \longrightarrow 0 \\
& & \parallel & & \parallel & & \parallel \\
& & H_{\mathbb{T}}^*(F_{c_i}) & & \bigoplus_{l=1}^i H_{\mathbb{T}}^*(F_{c_l}) & & \bigoplus_{l=1}^{i-1} H_{\mathbb{T}}^*(F_{c_l})
\end{array} \tag{4.1}$$

The top sequence is exact by Proposition 4.7, and the second is clearly exact. The vertical maps are restriction maps and we prove that they are injective by induction. The base case of the induction is trivial since  $M_{a_1}^-$  is empty. Now, the last column can be identified with  $H_{\mathbb{T}}^*(M_{b_{i-1}}^-) \rightarrow H_{\mathbb{T}}^*(F \cap M_{b_{i-1}}^-)$  by using  $H_{\mathbb{T}}^*(M_{b_{i-1}}^-) \cong_{\text{homot}} H_{\mathbb{T}}^*(M_{a_i}^-)$ . The map  $\gamma_i$  is injective because it is the map in the Proposition 4.7,

$$\begin{array}{ccc}
H_{\mathbb{T}}^i(M_b^-, M_a^-) & \longrightarrow & H_{\mathbb{T}}^i(M_b^-) \\
\cong_{MB} \downarrow & \searrow \gamma & \downarrow \\
H_{\mathbb{T}}^i(D_c, S_c) & & H_{\mathbb{T}}^i(D_c) \\
\cong_{Thom} \downarrow & \searrow & \downarrow \cong_{Homot} \\
H_{\mathbb{T}}^{i-\lambda}(F_c) & \xrightarrow{\cup e_{\mathbb{T}}(E_c^-)} & H_{\mathbb{T}}^i(F_c)
\end{array} . \tag{4.2}$$

The map  $\beta_i$  in Diagram (4.1) is injective by the induction hypothesis. Hence, by the Five Lemma, the middle map is injective.

If (Z3) is satisfied, then Proposition 4.7 holds with  $\mathbb{Z}$  coefficients, and the above argument carries through with  $\mathbb{Z}$  coefficients. This completes the proof.  $\square$

*Remark 4.11.* We will show that the toric orbifolds with the maximal torus  $\mathbb{R}$ -action satisfies the condition (Z3), and so Theorem 4.10 holds over  $\mathbb{Z}$ . Following Example 4.5, if we let  $F$  be any manifold with 2-torsion in its integral cohomology, and let  $S^1$  act on  $F \times \mathbb{C}P^1$  by fixing  $F$  pointwise and rotating  $\mathbb{C}P^1$  at double speed, then (Z3) fails, as in Example 4.5, a certain Euler class will be a zero-divisor and

$$i^* : H_{S^1}^*(F \times \mathbb{C}P^1; \mathbb{Z}) \rightarrow H_{S^1}^*((F \times \mathbb{C}P^1)^{S^1}; \mathbb{Z})$$

is not injective. Indeed, if  $\alpha$  is any 2-torsion class in  $H^*(F; \mathbb{Z})$ , then

$$\alpha \otimes 1 \in H^*(F; \mathbb{Z}) \otimes H_{S^1}^*(\mathbb{C}P^1; \mathbb{Z}) \cong H_{S^1}^*(F \times \mathbb{C}P^1; \mathbb{Z})$$

is in the kernel of  $i^*$ .

## 5. GENERALIZING TOLMAN AND WEITSMAN'S PROOF OF THE GKM THEOREM

We need the following technical lemma, which generalizes Lemma 3.2. [TW].

**Lemma 5.1.** *Let  $E$  be a  $\mathbb{T}$ -equivariant complex vector bundle over a manifold  $F$  with an action of  $\mathbb{T}$ . Assume that  $[F/\mathbb{S}]$  is an orbifold and  $\forall x \in F$ ,  $\mathbb{T} = \mathbb{T}_x \cdot \mathbb{S}$ . Assume that  $\mathbb{T}_{x,1}$  is independent of  $x \in F$ . Let  $E := \bigoplus_{\alpha} E_{\alpha}$  be the weight decomposition of the  $\mathbb{T}_{x,1}$ -action. Then over  $\mathbb{Q}$ , for each  $\eta \in H_{\mathbb{T}}^*(F)$ , we have*

*If  $\eta$  is a multiple of  $\mathbf{e}_{\mathbb{T}}(E_{\alpha})$  for all  $\alpha$ , then  $\eta$  is a multiple of  $\cup_{\alpha} \mathbf{e}_{\mathbb{T}}(E_{\alpha}) = \mathbf{e}_{\mathbb{T}}(E)$ .*

*If we assume  $\mathbb{T}_x$  is connected for all  $x \in F$ , this statement also holds over  $\mathbb{Z}$ .*

*Proof.* First we prove in the case when  $[F/\mathbb{S}]$  is 0-dimensional. Over  $\mathbb{Q}$ , we have the sequence of isomorphisms,

$$\begin{aligned} H_{\mathbb{T}}^*(F) &= H^*(E\mathbb{T} \times_{\mathbb{T}} F) \\ &= H^*(B\mathbb{T}_x \times E(\mathbb{S}/\mathbb{S}_x) \times_{\mathbb{S}/\mathbb{S}_x} F) \\ &\cong H^*(B\mathbb{T}_x) \\ &\cong H^*(B\mathbb{T}_{x,1}) \\ &= \text{Sym}\left(\text{Hom}(\mathbb{T}_{x,1}, S^1) \otimes \mathbb{Q}\right) \\ &= \mathbb{Q}[\alpha_1, \dots, \alpha_n]. \end{aligned}$$

The equality in the second line holds because  $\mathbb{S}/\mathbb{S}_x$  acts freely on  $F$ . Via this identification,  $\mathbf{e}_{\mathbb{T}}(E_{\alpha}) = \alpha^{n_{\alpha}}$  where  $n_{\alpha}$  is the rank of  $E_{\alpha}$ . Since  $\alpha$ 's are all distinct and non-zero, all  $\mathbf{e}_{\mathbb{T}}(E_{\alpha})$ 's are pairwise relatively prime. Hence  $\eta$  must be a multiple of  $\cup_{\alpha} \mathbf{e}_{\mathbb{T}}(E_{\alpha})$  since  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$  is a unique factorization domain over  $\mathbb{Q}$ . Over  $\mathbb{Z}$ , the same argument works as long as we have  $H_{\mathbb{T}}(F, \mathbb{Z}) \cong \text{Sym}\left(\text{Hom}(\mathbb{T}_{x,1}, S^1)\right) = \mathbb{Z}[\alpha_1, \dots, \alpha_n]$ . This happens precisely when  $\mathbb{T}_x$  is connected.

When  $[F/\mathbb{S}]$  is not 0-dimensional, we still have, over  $\mathbb{Q}$ ,

$$H_{\mathbb{T}}(F) = H(E\mathbb{T} \times_{\mathbb{T}} F) = H(B\mathbb{T}_{x,1} \times E(\mathbb{S}/\mathbb{S}_{x,1}) \times_{\mathbb{S}/\mathbb{S}_{x,1}} F) \cong H_{\mathbb{T}_{x,1}}(pt) \otimes H(F/\mathbb{S}).$$

Define the  $F$ -degree by  $H_{\mathbb{T}}(F) = \bigoplus_i H_{\mathbb{T}_{x,1}}(pt) \otimes H^i(F/\mathbb{S})$ ,  $a = \sum_i a_i$ . Then  $\mathbf{e}_{\mathbb{T}}(E_{\alpha})_0 = \alpha^{n_{\alpha}}$  since the class is determined by pulling back  $E_{\alpha}$  via  $\{x\} \hookrightarrow F$ . The remainder of the proof is purely algebraic and so the argument in [TW, p. 8] can be followed verbatim.

Over  $\mathbb{Z}$ , the above arguments works if  $H_{\mathbb{T}}(F) \cong H_{\mathbb{T}_{x,1}}(pt) \otimes H(F/\mathbb{S})$ . This happens exactly if  $\mathbb{T}_x$  is connected for all  $x \in F$ .  $\square$

Let  $[M/\mathbb{S}]$  be a compact, connected Hamiltonian  $\mathbb{R}$ -orbifold where  $\mathbb{S} \hookrightarrow \mathbb{T} \twoheadrightarrow \mathbb{R}$  is the torus extension such that  $\mathbb{T}$  acts on  $M$  extending the  $\mathbb{S}$ -action. The corresponding maps of Lie algebras are  $\mathfrak{s} \hookrightarrow \mathfrak{t} \twoheadrightarrow \mathfrak{r}$ . Let  $i : F \hookrightarrow M$  and  $j : F \hookrightarrow M_1$  be the inclusion maps.

*Remark 5.2.* Let  $N^\circ$  be a connected component of  $M_1^\circ$ , and let  $N$  be its closure. Then  $[N/S]$  is a Hamiltonian  $\mathbf{R}$ -orbifold with the moment map induced by the map  $\mu|_N : N \rightarrow \mathfrak{t}^*$ , the restriction of the original  $\mu : M \rightarrow \mathfrak{t}^*$  (see Remark 3.7). If the injectivity theorem holds over  $\mathbb{Q}$  (resp. over  $\mathbb{Z}$ ) for  $M$ , then it also holds over  $\mathbb{Q}$  (resp. over  $\mathbb{Z}$ ) for  $N$ . This is because the conditions (Q2) (resp. (Z2)) for  $M$  imply the ones for  $N$ .

We now turn to the constraints on a class restricted to the fixed point set. Using a component of the moment map to order the critical values, at the first fixed set where a class is non-zero, its restriction must be a multiple of the equivariant Euler class of the negative normal bundle.

**Proposition 5.3.** *Choose a generic  $\xi \in \mathfrak{t}$  and let  $F_c$  be a connected component of the intersection  $F \cap f^{-1}(c)$ , where  $f := \mu^\xi$  and  $c \in \mathbb{R}$  is a critical value for  $\mu^\xi$ . Let  $(a, b)$  be an open interval containing the single critical value  $c$ . Let  $M_{1,b}^- := M_1 \cap f^{-1}(-\infty, b)$  and  $F_a^- := F \cap f^{-1}(-\infty, a)$ . Then if  $\eta \in H_{\mathbf{T}}^*(M_{1,b}^-, \mathbb{Q})$  satisfies  $\eta|_{F_a^-} = 0 \in H_{\mathbf{T}}^*(F_a^-, \mathbb{Q})$ , we must have that  $\eta|_{F_c}$  is a multiple of  $\mathbf{e}_{\mathbf{T}}(E_c^-) \in H_{\mathbf{T}}^*(F_c, \mathbb{Q})$ , where  $E_c^-$  is the  $\mathbf{T}$ -equivariant negative normal bundle of  $F_c$  in  $M$ .*

*Moreover, the claim holds with integer coefficients if*

- (Z4) *the isotropy group  $\mathbf{T}_x$  is connected for all  $x \in F$ , and each weight of the  $\mathbf{T}_x$ -action on the normal bundle to  $F$  at  $x$  is primitive for all  $x \in F$  (i.e. (Z3) is satisfied).*

*Remark 5.4.* The hypothesis that  $\mathbf{T}_x$  be connected is a natural one. This hypothesis may be exploited to extend a variety of rational cohomology results to integral cohomology in equivariant symplectic geometry. This is discussed in work in progress by the first author and Tolman [HT]. This hypothesis also arises in the work of Franz and Puppe [FP, Theorem 1.1; and Examples 5.3 and 5.4].

*Proof.* There is a connected component  $N^\circ$  of  $M_1^\circ$  such that its closure  $N$  contains  $F_c$ , since if  $F_c \cap N \neq \emptyset$ , then  $F_c \subset N$  by Lemma 3.6. If  $\eta|_{F_a^-} = 0$ , then  $\eta|_{N \cap F_a^-} = \eta|_{F \cap N_a^-} = 0$ . In the proof of Theorem 4.10 for the  $\mathbf{R}$ -action on  $[N/S]$ , in the middle of the induction step we have the injection  $H_{\mathbf{T}}^*(N_a^-) \hookrightarrow H_{\mathbf{T}}^*(F \cap N_a^-)$ . Hence  $\eta|_{F \cap N_a^-} = 0$  implies  $\eta|_{N_a^-} = 0$ . Apply Proposition 4.7 to the  $\mathbf{T}$ -action on  $N$ : we obtain a short exact sequence and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathbf{T}}^i(N_b^-, N_a^-) & \longrightarrow & H_{\mathbf{T}}^i(N_b^-) & \xrightarrow{\beta} & H_{\mathbf{T}}^i(N_a^-) \longrightarrow 0 \\ & & \parallel & & \downarrow \text{restriction} & & \\ & & H_{\mathbf{T}}^{i-\lambda}(F_c) & \xrightarrow{\cup \mathbf{e}_{\mathbf{T}}(E_{N,c}^-)} & H_{\mathbf{T}}^i(F_c) & & \end{array}$$

where  $E_{N,c}^-$  is the  $\mathbf{T}$ -equivariant negative normal bundle of  $F_c$  in  $N$ . The exactness and the commutativity of those diagram implies that any element in the kernel  $\ker \beta$  is a multiple of the equivariant Euler class  $\mathbf{e}_{\mathbf{T}}(E_{N,c}^-)$  when it is restricted to  $F_c$ . Thus by Lemma



5.1 above, we are done.  $\square$

We have now established the preliminaries necessary to prove the orbifold version of Tolman and Weitsman's version of the GKM theorem.

**Theorem 5.5.** *In the diagrams*

$$\begin{array}{ccc} M & & M_1 \\ \uparrow i & \nearrow j & \\ F & & \end{array} \quad \xRightarrow{\text{take } H_T^*} \quad \begin{array}{ccc} H_T(M) & & H_T(M_1) \\ \downarrow i^* & \nwarrow j^* & \\ H_T(F) & & \end{array},$$

we have  $\text{Im } i^* = \text{Im } j^*$  over  $\mathbb{Q}$ . In particular,  $H_T^*(M) \cong \text{Im } j^*$ . If we assume that  $T_x$  is connected for all  $x \in F$  and that the weights of the  $T_{x,1}$ -action on the negative normal bundle (for a generic  $\mu^\xi$ ) are primitive for all  $x \in F$ , the claim also holds over  $\mathbb{Z}$ .

*Proof.* We have proved all the preliminary claims needed, so the proof goes through exactly as in [TW, p. 8–9]. We proceed by induction on the index  $i = 1, \dots, n$ , where the critical values  $c_i$  are ordered so that  $c_1 < \dots < c_n$ . Let  $(a, b)$  be an open interval containing only  $c_i$ . Consider the following map of short exact sequences, using the notation from Proposition 5.3:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_T^*(M_b^-, M_a^-) & \longrightarrow & H_T^*(M_b^-) & \longrightarrow & H_T^*(M_a^-) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker r^*|_{\text{Im } j_b^-} & \longrightarrow & \text{Im } j_b^- & \longrightarrow & \text{Im } j_a^- \longrightarrow 0 \end{array},$$

where  $r : M_{1,a}^- \rightarrow M_{1,b}^-$  is the obvious inclusion. The third vertical map is surjective by the inductive assumption. The surjectivity of the first vertical map follows from Proposition 5.2. The surjectivity of the middle vertical map then follows by a diagram chase analogous to the one in the Five Lemma.  $\square$

## 6. GKM COMPUTATIONS FOR TORIC ORBIFOLDS

In this section, we compute the  $\mathbb{R}$ -equivariant cohomology of  $[M/\mathbf{S}]$  for compact symplectic toric orbifolds. We show that it is isomorphic, with  $\mathbb{Z}$ -coefficients, to the Stanley-Reisner ring of the corresponding moment polytope. This generalizes the case of smooth toric manifolds. This result should also follow from an argument using the moment angle complex, since the moment angle complex only depends on the combinatorial type of the polytope (see [BP], [P]). Nonetheless, we will now apply our techniques to obtain the following.

**Theorem 6.1.** *Let  $\mathbf{S}$  be an  $(m - n)$ -dimensional subtorus of the  $m$ -dimensional torus  $\mathbf{T}$  which acts on  $\mathbb{C}^m$  coordinate-wise, and let  $\mu_{\mathbf{S}} : \mathbb{C}^m \rightarrow \mathfrak{s}^*$  be the induced moment map of the  $\mathbf{S}$ -action, so that  $M := \mu_{\mathbf{S}}^{-1}(\eta)$  is a compact manifold, for a regular value  $\eta \in \mathfrak{s}^*$ .*

Let  $\Delta$  be the moment polytope of the compact toric orbifold obtained as the symplectic reduction  $\mathbb{C}^m // \mathbf{S} = [M/\mathbf{S}]$  at the regular value  $\eta$ . Then

$$H_{\mathbf{R}}^*(\mathbb{C}^m // \mathbf{S}, \mathbb{Z}) \cong \text{SR}(\Delta)$$

where  $\text{SR}(\Delta)$  is the Stanley-Reisner ring of the polytope  $\Delta$ .

The (ordinary) Chow of an algebraic toric orbifold has been computed by Iwanari [I] as the quotient of the Stanley-Reisner ring by linear terms. The ordinary integral cohomology  $H^*(X; \mathbb{Z})$  need not be the quotient of the Stanley-Reisner ring by linear terms. As discussed in [H] after the proof of Theorem 4.2, the integral cohomology of a direct product of two identical weighted projective spaces has torsion in odd degrees, whereas the Stanley-Reisner ring can only contribute to even degrees. This is explored further in [LMM].

**6.1. Stanley Reisner ring and the direct sum decomposition.** Let  $\Delta$  be a simple polytope of  $n$ -dimension. We let  $K_{\Delta}$  denote the associated simplicial complex whose vertices are the facets of  $\Delta$ , and a collection of vertices is a simplex in  $K_{\Delta}$  if and only if the corresponding collection of facets in  $\Delta$  has non-empty intersection. The Stanley-Reisner ring  $\text{SR}(\Delta)$  of  $\Delta$  is defined as the Stanley-Reisner ring  $\text{SR}(K_{\Delta})$  of the simplicial complex  $K_{\Delta}$  associated to  $\Delta$ . Namely,

$$\text{SR}(\Delta) := \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle \prod_{i \in \sigma} x_i \mid \sigma \notin K_{\Delta} \rangle}$$

where  $m$  is the number of facets of  $\Delta$ . We define the ring

$$\begin{aligned} R &:= \bigoplus_{v \text{ vertex of } \Delta} \mathbb{Z}[x_{i_1}, \dots, x_{i_n} \mid v = H_{i_1} \cap \dots \cap H_{i_n}] \\ &= \bigoplus_{v \text{ vertex of } \Delta} \mathbb{Z}[x_{i_1}, \dots, x_{i_n} \mid H_v^* = \{i_1, \dots, i_n\}], \end{aligned}$$

where  $\{H_1, \dots, H_m\}$  is the set of facets of  $\Delta$  and  $H_v^*$  is the facet of  $K_{\Delta}$  corresponding to the vertex  $v$  of  $\Delta$ . We define a subring of  $R$  by

$$R_{\Delta} := \left\{ (p_{v_1}, \dots, p_{v_l}) \in R \mid \begin{array}{l} p_v|_{x_j=0} = p_{v'}|_{x_i=0} \\ \forall \text{ edge } (v, v') \text{ in } \Delta, \\ \text{where } j \in H_v^* \setminus H_{v'}^*, \ i \in H_{v'}^* \setminus H_v^* \end{array} \right\}.$$

This ring  $R_{\Delta}$  is the algebra of continuous piece-wise polynomial functions on the fan canonically defined by the simplicial complex  $K_{\Delta}$  and is well-known to be isomorphic to  $\text{SR}(\Delta)$  via

$$x_i \mapsto (p_{v_1}^i, \dots, p_{v_l}^i), \text{ where } p_v^i = \begin{cases} 0 & \text{if } i \notin H_v^* \\ x_i & \text{if } i \in H_v^* \end{cases}.$$

See, for example, [Br, §1.3], [BR] or [Bl].

**6.2. Injectivity over  $\mathbb{Z}$  for toric orbifolds.** We recall the construction of symplectic toric orbifolds in [LT]. Let  $\Delta$  be a simple integral polytope in  $\mathbb{R}^n$ . We may identify  $\mathbb{R}^n \cong \mathfrak{r}^*$ . Let  $\mathcal{H} = \{H_1, \dots, H_m\}$  be the set of facets and  $\rho_1, \dots, \rho_m \in \mathbb{Z}^n$  the primitive inward normal vectors to facets and let  $b_1, \dots, b_m \in \mathbb{Z}_{>0}$  be positive integers that label the facets of  $\Delta$  in the sense of [LT]. The polytope is given by

$$\Delta = \left\{ v \in \mathbb{R}^n \mid \langle b_i \rho_i, v \rangle + \eta_i \geq 0, \quad i = 1, \dots, m \right\}$$

for some  $\eta := (\eta_i) \in \mathbb{R}^n$ . Let  $B := [b_1 \rho_1 \mid \dots \mid b_m \rho_m] \in \text{Mat}_{n \times m}(\mathbb{Z})$ . Suppose that the transpose  ${}^t B : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$  has free cokernel so that we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{m-n} \xrightarrow{A} \mathbb{Z}^m \xrightarrow{B} \mathbb{Z}^n \longrightarrow 0.$$

The matrix  $A \in \text{Mat}_{m, m-n}(\mathbb{Z})$  is given by choosing a basis of  $\ker(B)$ . Now applying  $\text{Hom}(-, S^1)$ , we get an exact sequence of tori

$$1 \longrightarrow \mathbf{S}^{(m-n)} \xrightarrow{\tilde{A}} \mathbf{T}^{(m)} \xrightarrow{\tilde{B}} \mathbf{R}^{(n)} \longrightarrow 1,$$

and the corresponding exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{s}^{(m-n)} \xrightarrow{A} \mathfrak{t}^{(m)} \xrightarrow{B} \mathfrak{r}^{(n)} \longrightarrow 0.$$

Let  $\bar{\mu} : \mathbb{C}^m \rightarrow \mathfrak{s}^*$  be the moment map for the action of  $\mathbf{S}$  defined by the standard  $\mathbf{T}$ -action on  $\mathbb{C}^m$  through the exact sequence above. This sends  $(z_1, \dots, z_m) \in \mathbb{C}^m$  to

$$z = (z_1, \dots, z_m) \mapsto {}^t A \cdot \begin{bmatrix} |z_1|^2 \\ \vdots \\ |z_m|^2 \end{bmatrix} = \begin{bmatrix} a_{11}|z_1|^2 + \dots + a_{m1}|z_m|^2 \\ \vdots \\ a_{1, m-n}|z_1|^2 + \dots + a_{m, m-n}|z_m|^2 \end{bmatrix},$$

where  ${}^t A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m-n}}$ . The orbifold  $[M/\mathbf{S}^{(m-n)}]$  corresponding to the labeled polytope  $(\Delta, b)$  is given by reduction at  $\eta' := {}^t A \cdot \eta \in \mathfrak{s}^*$ . Namely,

$$M = \mu^{-1}(\eta') = \left\{ z \in \mathbb{C}^m \mid {}^t A \cdot |z|^2 = \eta' \right\}.$$

**Lemma 6.2** ([LT], proof of Theorem 8.1). *Let  $v = H_{i_1} \cap \dots \cap H_{i_n} \in \mathfrak{r}^* \cong \mathbb{R}^n$  be a vertex of  $\Delta$ . Then the corresponding fixed orbifold point in  $[M/\mathbf{S}]$  is given by  $[F_v/\mathbf{S}]$  where*

$$F_v = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid |z_i|^2 = ({}^t B \cdot v + \eta)_i, \quad i = 1, \dots, m\},$$

*and  $({}^t B \cdot v + \eta)_i = 0$  if and only if  $i = i_1, \dots, i_n$ . For each  $x \in F_v$ , the isotropy group  $\mathbf{T}_x$  is a subtorus of  $\mathbf{T}$*

$$\mathbf{T}_x = \left\{ (t_1, \dots, t_m) \in \mathbf{T} \mid t_i = 1 \quad \forall i \in [m] \setminus \{i_1, \dots, i_n\} \right\}.$$

*In particular, the isotropy  $\mathbf{T}_{F_v}$  of  $F_v$  equals  $\mathbf{T}_x$  for every  $x \in F_v$ .*

*Proof.* This follows from the proof of Theorem 8.1 in [LT]. Since a vertex  $v \in \Delta \subset \mathbb{R}^n$  is given by  $n$  equations  $\langle b_{i_1} \rho_{i_1}, v \rangle + \eta_{i_1} = 0, \dots, \langle b_{i_n} \rho_{i_n}, v \rangle + \eta_{i_n} = 0$ , we have  $({}^t B \cdot v + \eta)_i = 0$  if and only if  $i = i_1, \dots, i_n$ .  $\square$

The local normal form around a point  $p_v$  in  $F_v$  is given by

$$U_{p_v} \cong \mathbb{T} \times_{\mathbb{T}_{F_v}} W, \text{ where } W \cong \mathbb{C} \cdot \frac{\partial}{\partial z_{i_1}} \oplus \cdots \oplus \mathbb{C} \cdot \frac{\partial}{\partial z_{i_n}}$$

and the weight of  $\mathbb{C} \cdot \frac{\partial}{\partial z_{i_k}}, k = 1, \dots, n$  is  $\lambda_{i_k} \in \text{Hom}(\mathbb{T}_{F_v}, S^1)$  defined by  $\lambda_{i_k}(t) = t_{i_k}$  for all  $t \in \mathbb{T}_{F_v}$ . Therefore, the action satisfies hypothesis (Z2). Thus Theorem 5.5 holds over  $\mathbb{Z}$ .

**Proposition 6.3.** *The restriction map  $H_{\mathbb{T}}^*(M, \mathbb{Z}) \rightarrow H_{\mathbb{T}}^*(F, \mathbb{Z})$  is injective where  $F$  is the union of  $F_v$ 's for all vertices  $v \in \Delta$  and its image coincides with the image of  $H_{\mathbb{T}}^*(M_1, \mathbb{Z}) \rightarrow H_{\mathbb{T}}^*(F, \mathbb{Z})$ .*

**6.3. The 1-skeleton and GKM computations.** Recall that  $H_v^*$  is the facet of the associated simplicial complex  $K_{\Delta}$  corresponding to a vertex  $v$  of  $\Delta$ . For each edge  $(v, u)$  of  $\Delta$ , we have  $|H_v^* \cap H_u^*| = n - 1$ . Letting  $\{a\} = H_v^* \setminus H_u^*$  and  $\{b\} = H_u^* \setminus H_v^*$ , the corresponding component of the 1-skeleton is

$$N_{v,u} = \left\{ (z_1, \dots, z_m) \in \mathbb{C}^m \mid |z_k|^2 = s\bar{v}_k + (1-s)\bar{u}_k, s \in [0, 1] \right\},$$

where  $\bar{v} = {}^t B \cdot v + \eta$ . Therefore,  $(z_1, \dots, z_m) \in N_{v,u}$  is given by

$$\begin{aligned} |z_k|^2 &= 0 & \forall k \in H_v^* \cap H_u^*, \\ |z_b|^2 &= s\bar{v}_b \neq 0, \\ |z_a|^2 &= (1-s)\bar{u}_a \neq 0, \\ |z_l|^2 &= s\bar{v}_l + (1-s)\bar{u}_l \neq 0 & \forall l \notin H_v^* \cup H_u^*. \end{aligned}$$

By getting rid of the parameter  $s$ , we can also write  $N_{u,v}$  as

$$\begin{aligned} |z_k|^2 &= 0 & \forall k \in H_v^* \cap H_u^*, \\ \bar{u}_a |z_b|^2 + \bar{v}_b |z_a|^2 &= 2\bar{u}_a \bar{v}_b \neq 0, \\ |z_l|^2 &= \frac{|z_b|^2}{\bar{v}_b} \bar{v}_l + \left(1 - \frac{|z_b|^2}{\bar{v}_b}\right) \bar{u}_l \neq 0 & \forall l \notin H_v^* \cup H_u^*. \end{aligned}$$

The pair  $(\mathbb{C}^m, N_{v,u})$  is  $\mathbb{T}$ -equivariantly homotopic to the pair

$$(\mathbb{C}^m, \{0\}^{n-1} \times S^3 \times (S^1)^{m-n-1}).$$

We have the following short exact sequence (cf. Theorem 4.2 [H])

$$0 \rightarrow H_{(S^1)^k}^i(\mathbb{C}^k, S^{2k-1}) \rightarrow H_{(S^1)^k}^i(\mathbb{C}^k) \rightarrow H_{(S^1)^k}^i(S^{2k-1}) \rightarrow 0.$$

By applying the Künneth formula, we obtain a surjection  $H_{\mathbb{T}}^*(\mathbb{C}^m) \twoheadrightarrow H_{\mathbb{T}}^*(N_{v,u})$ .

**Lemma 6.4.** *The inclusion  $\pi' : N_{v,u} \hookrightarrow \mathbb{C}^m$  induces a surjective map*

$$\pi'^* : H_{\mathbb{T}}^*(\mathbb{C}^m) \rightarrow H_{\mathbb{T}}^*(N_{v,u}).$$

Furthermore, since  $\mathbb{C}^m$  equivariantly retracts to  $\{0\}$ , by the commutativity of the diagram

$$\begin{array}{ccc} & & ET \times_T \mathbb{C}^m \\ & \nearrow \pi' & \parallel \text{homot} \\ ET \times_T N_{v,u} & \xrightarrow{\pi} & BT \end{array},$$

$\pi^*$  is surjective.

Now consider the diagram

$$\begin{array}{ccc} ET \times_T (N_{v,u}) & \xrightarrow{\pi} & BT \\ \uparrow i & \nearrow \rho & \\ ET \times_T F_v \sqcup ET \times_T F_u & & \end{array}.$$

Taking cohomology with  $\mathbb{Z}$  coefficients, we obtain

$$\begin{array}{ccc} H_T(N_{v,u}) & \xleftarrow{\pi^*} & \mathbb{Z}[x_1, \dots, x_m] \\ \downarrow i^* & \nwarrow \rho^* & \\ \mathbb{Z}[x_{i_1}, \dots, x_{i_{n-1}}, x_a] \oplus \mathbb{Z}[x_{i_1}, \dots, x_{i_{n-1}}, x_b] & & \end{array},$$

where  $H_v^* = \{i_1, \dots, i_{n-1}, a\}$  and  $H_u^* = \{i_1, \dots, i_{n-1}, b\}$ . The image of  $i^*$  consists  $(p_v, p_u)$  such that  $p_v|_{x_a=0} = p_u|_{x_b=0}$ . Thus the image of  $H_T^*(M_1)$  in  $H_T(F)$  is

$$\left\{ (p_{v_1}, \dots, p_{v_l}) \in R \mid \begin{array}{l} p_v|_{x_j=0} = p_{v'}|_{x_j=0} \\ \forall \text{ edge } (v, v') \text{ in } \Delta \\ \text{where } \{j\} = H_v^* \setminus H_{v'}^* \quad \{i\} = H_{v'}^* \setminus H_v^* \end{array} \right\},$$

which coincides with  $R_\Delta$  defined in Section 6.1. Thus we may conclude that the R-equivariant cohomology of the toric orbifold  $[M/S]$  is isomorphic to the Stanley-Reisner ring of  $\Delta$ . This completes the proof of Theorem 6.1.

## 7. R-EQUIVARIANT CHEN-RUAN ORBIFOLD COHOMOLOGY OF $[M/S]$

We now turn to R-equivariant Chen-Ruan theory. When R is the trivial group, our definitions agree with the usual non-equivariant ones [CR]. After defining the R-equivariant Chen-Ruan orbifold cohomology ring of the R-orbifold  $[M/S]$ , we survey the literature on this topic which motivates our definitions and results. The inertia manifold for the locally free S-action on  $M$  is defined by  $\mathcal{I}_S M := \bigsqcup_{g \in S} M^g$ , where  $M^g$  is the set of fixed points by the subgroup  $\langle g \rangle$  generated by  $g$ . This disjoint union is a finite union, since  $M$  is compact and the S-action is locally free. There is also an induced T-action on  $M$ , which allows us to define the Hamiltonian R action on the orbifold

$$\mathcal{I}[M/S] := \bigsqcup_{g \in S} [M^g/S],$$

called the **inertia orbifold**. We may then define the **R-equivariant Chen-Ruan cohomology** of  $[M/S]$ , as a vector space, to be

$$H_{orb,R}([M/S]) := H_T(\mathcal{I}_S M) = \bigoplus_{g \in S} H_T(M^g) = \bigoplus_{g \in S} H_R([M^g/S]).$$

Let  $M^{g,h} := M^g \cap M^h$ . The normal bundle  $N_{M^g \subset M}$  of  $M^g$  in  $M$  is then a  $T$ -equivariant complex vector bundle, with weight decomposition

$$N_{M^g \subset M} = \bigoplus_{\lambda \in \text{Hom}(\langle g \rangle, S^1)} W_\lambda.$$

Define an element  $\mathcal{S}_g$  of the  $T$ -equivariant (topological)  $K$ -theory  $K_T(M^g) \otimes \mathbb{Q}$  of  $M^{g,h}$  over  $\mathbb{Q}$  by

$$\mathcal{S}_g = \bigoplus_{\lambda \in \text{Hom}(\langle g \rangle, S^1)} a_\lambda(g) W_\lambda$$

where  $a_\lambda(g) \in [0, 1)$  is the **age**, defined by  $\lambda(g) = e^{2\pi i a_\lambda(g)}$ . Following [EJK, JKK], define the **equivariant virtual bundle**  $\mathcal{R}_M(g, h)$  as an element of  $K_T(M^{g,h}) \otimes \mathbb{Q}$

$$\mathcal{R}_M(g, h) := \ominus N_{M^{g,h} \subset M} \oplus \mathcal{S}_g|_{M^{g,h}} \oplus \mathcal{S}_h|_{M^{g,h}} \oplus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h}}.$$

Since  $H := \langle g, h \rangle$  acts on each tangent space  $T_x M$  and  $T_x M / T_x M^{g,h}$  for  $x \in M^{g,h}$ , we have the decomposition

$$N_{M^{g,h} \subset M} = \bigoplus_{\lambda \in \text{Hom}(H, S^1)} W_\lambda.$$

Since the  $T$ -action commutes with the  $H$ -action, this decomposition is  $T$ -stable; that is, each  $W_\lambda$  is a  $T$ -equivariant complex vector bundle. Then we may show that

$$\mathcal{R}_M(g, h) = \bigoplus_{\substack{a_\lambda(g) + a_\lambda(h) + a_\lambda((gh)^{-1}) = 2, \\ \lambda \neq 0}} W_\lambda.$$

Thus  $\mathcal{R}_M(g, h)$  is actually represented by a  $T$ -equivariant complex vector bundle. This is the version of the obstruction bundle introduced in [BCS]. Since  $\mathcal{R}(g, h)$  is a  $T$ -equivariant complex vector bundle on  $M^{g,h}$ , we take the  $T$ -equivariant Euler class to define

$$c_M(g, h) := \mathbf{e}_T(\mathcal{R}_M(g, h)) \in H_T(M^{g,h}),$$

called the **virtual class**. We define the **R-equivariant orbifold product** on  $H_{orb,R}([M/S])$  by the usual pull-cup-push formula. Namely for  $\eta \in H_T(M^g)$  and  $\xi \in H_T(M^h)$ ,

$$\eta \odot \xi := e_* (e_1^* \eta \cup e_2^* \xi \cup c_M(g, h)), \quad (7.1)$$

where  $e_1, e_2$ , and  $e$  are the obvious inclusions of  $M^{g,h}$  into  $M^g, M^h$ , and  $M^{gh}$  respectively. For the definition of the equivariant pushforward  $e_*$ , see for example [AB] Section 2. The associativity of this product follows immediately from the proof of the corresponding associativity in non-equivariant case in [BCS, GHK, JKK]. For  $\eta \in H^{|\eta|}(M^g)$ , the rational grading is assigned by  $\deg_{\mathbb{Q}} \eta = |\eta| + 2 \cdot \text{age}(g)$ , where  $\text{age}(g) := \text{rank } \mathcal{S}_g$ . It

follows immediately that the product is rationally graded. If one of  $g$ ,  $h$ , or  $gh$  is the identity, then the obstruction bundle has rank 0 and so it is easy to see that  $H_T^*(M)$  sits in  $H_{orb,R}^*([M/S])$  as a subalgebra. In particular,  $H_{orb,R}^*([M/S])$  is a  $H_T^*(pt)$ -algebra.

**Theorem 7.1.**  *$H_{orb,R}^*([M/S])$  is a rationally graded, associative,  $H_T^*(pt)$ -algebra.*

The usual Chen-Ruan orbifold cohomology groups of an algebraic orbifold  $\mathcal{X}$  are defined as a vector space by  $H^*(\mathcal{IX})$ , where  $\mathcal{IX}$  is the inertia orbifold. The orbifold product is the usual cup product which is then deformed by the Euler class of the obstruction bundle for the corresponding Gromov-Witten theory [AGV, Section 6]. The obstruction bundle for algebraic toric orbifolds has been computed by [BCS] and adopted for symplectic orbifolds by [GHK] following the original definitions in [CR]. The most recent formula for algebraic orbifolds that are global quotients by algebraic groups can be found in [EJK].

If  $\mathcal{X}$  is an algebraic  $G$ -orbifold, there is an induced action on the obstruction bundle defined in [AGV], and the  $G$ -equivariant Chen-Ruan cohomology ring can be defined as the equivariant cohomology groups of the inertia orbifold together with the orbifold cup product deformed by the equivariant Euler class of the obstruction bundle [J, Section 2.2.1]. On the other hand, the obstruction bundle defined in [CR] for a symplectic  $G$ -orbifold is also naturally  $G$ -equivariant. In the case of the symplectic  $R$ -orbifolds considered in this paper, the formula in [GHK] derived from the definition in [CR] is  $R$ -equivariantly valid. The argument in [GH, Appendix A] can be made equivariant. It is also possible to verify that the computation done in [BCS] is valid  $R$ -equivariantly.

## 8. INJECTIVITY THEOREM FOR EQUIVARIANT CHEN-RUAN ORBIFOLD COHOMOLOGY

The ring  $H_{orb,R}^*([M/S])$  is not functorial: a map between spaces may not induce a map on Chen-Ruan orbifold cohomology rings. In particular, the inclusion of the fixed points  $[F/S] \hookrightarrow [M/S]$  does not induce a map in Chen-Ruan orbifold cohomology. Thus we introduce a new ring  $\mathcal{NH}_R^*(\nu[F/S])$ . Recall from Section 3.1 that

$$F := \{x \in M \mid T \cdot x = S \cdot x\}$$

is a submanifold of  $M$ , and the suborbifold  $[M/S]^R$  of  $R$ -fixed orbifold points of  $[M/S]$  is exactly  $[F/S]$ . The ring  $\mathcal{NH}_R^*(\nu[F/S])$  will be defined for the normal bundle of  $[F/S]$  in  $[M/S]$ . This new ring is defined only using the fixed points and isotropy data at the fixed points.

**Definition 8.1.** As a vector space, we define

$$\mathcal{NH}_R(\nu[F/S]) := \bigoplus_{g \in S} H_T(F^g).$$

The rational grading is defined with an age shift, exactly as in the previous section. We have the natural restriction map from  $H_{orb,R}([M/S])$  to  $\mathcal{NH}_R(\nu[F/S])$  and in order to make it a ring homomorphism, the product must be defined appropriately in

$\mathcal{NH}_R(\nu[F/S])$  using a push-cup-pull formula. Let  $\mathcal{E}(g, h)$  be the excess intersection bundle  $\mathcal{E}(g, h) = N_{M^{g,h} \subset M^{gh}}|_{F^{g,h}} \ominus N_{F^{g,h} \subset F^{gh}}$  for the diagram

$$\begin{array}{ccc} M^{g,h} & \longrightarrow & M^{gh} \\ \uparrow & \square & \uparrow \\ F^{g,h} & \longrightarrow & F^{gh} \end{array} .$$

Define

$$\mathcal{R}'_F(g, h) := \mathcal{R}_M(g, h)|_{F^{g,h}} \oplus \mathcal{E}(g, h), \quad c'_F(g, h) := \mathbf{e}_T(\mathcal{R}'_F(g, h)).$$

The product  $\star$  on  $\mathcal{NH}_R(\nu[F/S])$  is defined for  $a \in H_T(F^g)$  and  $b \in H_T(F^h)$  by  $a \star b := f_* (f_1^* a \cup f_2^* b \cup c'_F(g, h))$  where  $f_1, f_2, f$  are the obvious inclusions of  $F^{g,h}$  into  $F^g, F^h$  and  $F^{gh}$ .

**Theorem 8.2.**  $(\mathcal{NH}_R^*(\nu[F/S]), \star)$  is an associative graded ring.

*Proof.* Let  $g, h, m \in \mathbf{S}$ . Denote all relevant inclusions by

$$\begin{array}{ccccc} F^g & \xleftarrow{f_1} & F^{g,h} & \xrightarrow{\quad} & F^{gh} \\ & \searrow f_2 & \uparrow \phi & \square & \uparrow f_4 \\ F^h & & F^{g,h,m} & \xrightarrow{\psi} & F^{gh,m} \\ & & & & \downarrow l \\ & & & & F^{ghm} \\ & \swarrow f_3 & & & \\ F^m & & & & \end{array}$$

and

$$\begin{array}{ccccc} F^g & & & & \\ & \nwarrow \bar{f}_1 & & & \\ F^h & \xleftarrow{\bar{f}_2} & F^{g,h,m} & \xrightarrow{\psi} & F^{gh,m} \\ & \nwarrow \bar{f}_3 & & & \downarrow l \\ & & & & F^{ghm} \\ & \swarrow f_3 & & & \\ F^m & & & & \end{array} .$$



Let us calculate

$$\begin{aligned}
(a \star b) \star c &= l_*(f_4^* f_*(f_1^* a \cup f_2^* b \cup c'_F(g, h)) \cup f_3^* c \cup c'_F(gh, m)) \\
&\quad \text{by definition;} \\
&= l_*(\psi_*(\phi^*(f_1^* a \cup f_2^* b \cup c'_F(g, h)) \cup \epsilon) \cup f_3^* c \cup c'_F(gh, m)) \\
&\quad \text{by the excess intersection formula;} \\
&= l_*(\psi_*(\phi^* f_1^* a \cup \phi^* f_2^* b \cup \phi^* c'_F(g, h) \cup \epsilon) \cup f_3^* c \cup c'_F(gh, m)) \\
&\quad \text{because pull-back commutes with cup product;} \\
&= l_*(\psi_*(\bar{f}_1^* a \cup \bar{f}_2^* b \cup \phi^* c'_F(g, h) \cup \epsilon) \cup f_3^* c \cup c'_F(gh, m)) \\
&\quad \text{because } \bar{f}_1 = f_1 \circ \phi \text{ and } \bar{f}_2 = f_2 \circ \phi; \\
&= l_* \psi_*(\bar{f}_1^* a \cup \bar{f}_2^* b \cup \phi^* c'_F(g, h) \cup \epsilon \cup \psi^* f_3^* c \cup \psi^* c'_F(gh, m)) \\
&\quad \text{by the projection formula;} \\
&= l_* \psi_*(\bar{f}_1^* a \cup \bar{f}_2^* b \cup \bar{f}_3^* c \cup \phi^* c'_F(g, h) \cup \epsilon \cup \psi^* c'_F(gh, m)).
\end{aligned}$$

In the last line, we denote  $\epsilon := \mathbf{e}_T(E)$  where the bundle

$$E := N_{Fg,h \subset Fgh}|_{Fg,h,m} \ominus N_{Fg,h,m \subset Fgh,m}$$

is the excess intersection bundle corresponding to the square in above diagram. Now  $\phi^* c'_F(g, h) \cup \epsilon \cup \psi^* c'_F(gh, m)$  is the T-equivariant Euler class of

$$\begin{aligned}
&\phi^*(\mathcal{R}_M(g, h)|_{Fg,h} \oplus \mathcal{E}(g, h)) \oplus E \oplus \psi^*(\mathcal{R}_M(gh, m)|_{Fgh,m} \oplus \mathcal{E}(gh, m)) \\
&= \ominus N_{Fg,h,m \subset Fgh,m} \ominus N_{Mgh,m \subset M} \oplus \mathcal{S}_g \oplus \mathcal{S}_h \oplus \mathcal{S}_m \oplus \mathcal{S}_{(ghm)^{-1}},
\end{aligned}$$

where we omit  $|_{Fg,h,m}$  everywhere, and the only non-obvious cancellation uses

$$\mathcal{S}_g \oplus \mathcal{S}_{g^{-1}} = TM \ominus TM^g.$$

The final form is symmetric in  $(g, h, m)$ , establishing the associativity of  $\star$ .  $\square$

The following is an immediate generalization of the product to an  $n$ -fold product.

**Corollary 8.3.** *For  $a_i \in H_T(F^{g_i})$ ,  $i = 1, \dots, n$ , we may define an  $n$ -fold product by*

$$a_1 \star a_2 \star \dots \star a_n = \mathbf{f}_*(\bar{f}_1^* a_1 \cup \bar{f}_2^* a_2 \cup \dots \cup \bar{f}_n^* a_n \cup \mathbf{e}_T(\mathcal{R}'_F(g_1, \dots, g_n)))$$

where  $\bar{f}_i : F^{g_1, \dots, g_n} \rightarrow F^{g_i}$  and  $\mathbf{f} : F^{g_1, \dots, g_n} \rightarrow F^{\prod g_i}$  are obvious inclusions and the obstruction bundle is given by

$$\mathcal{R}'_F(g_1, \dots, g_n) := \ominus N_{F^{g_1, \dots, g_n} \subset F^{\prod g_i}} \ominus N_{M^{\prod g_i} \subset M} \oplus \bigoplus_{i=1}^n \mathcal{S}_{g_i} \oplus \mathcal{S}_{(\prod g_i)^{-1}}.$$

The inclusions  $i : F^g \rightarrow M^g$  for all  $g \in \mathbf{S}$  altogether induce a rationally graded linear map  $\mathcal{I} : H_{orb, \mathbf{R}}([M/\mathbf{S}]) \rightarrow \mathcal{NH}_{\mathbf{R}}(\nu[F/\mathbf{S}])$ . Consider the diagram of obvious inclusions:

$$\begin{array}{ccccc} M^g \times M^h & \xleftarrow{\Delta_M} & M^{g,h} & \xrightarrow{e} & M^{gh} \\ (i_1, i_2) \uparrow & & \uparrow j & \square & \uparrow i \\ F^g \times F^h & \xleftarrow{\Delta_F} & F^{g,h} & \xrightarrow{f} & F^{gh} \end{array}$$

The map  $\mathcal{I}$  is a graded ring homomorphism if

$$\begin{aligned} i^*(e_*(\Delta_M^*(\eta \otimes \xi) \cup c_{\mathbf{T}}(g, h))) &= f_*(\Delta_F^*(i_1, i_2)^*(\eta \otimes \xi) \cup c'_{\mathbf{T}}(g, h)) \\ &= f_*(j^*\Delta_M^*(\eta \otimes \xi) \cup c'_{\mathbf{T}}(g, h)). \end{aligned}$$

Since  $c'_{\mathbf{T}}(g, h) = j^*(c_{\mathbf{T}}(g, h)) \cup \mathbf{e}_{\mathbf{T}}(\mathcal{E}(g, h))$ , the equality follows exactly from the excess intersection formula. Thus, combined with the injectivity theorem, we obtain

**Theorem 8.4.** *The natural map  $\mathcal{I} : (H_{orb, \mathbf{R}}([M/\mathbf{S}]), \odot) \rightarrow (\mathcal{NH}_{\mathbf{R}}(\nu[F/\mathbf{S}]), \star)$  is a graded ring homomorphism. If the  $\mathbf{T}$ -action on  $M$  satisfies the condition (Q2) (resp. (Z2)), then this homomorphism is injective over  $\mathbb{Q}$  (resp. over  $\mathbb{Z}$ ).*

## 9. EXAMPLES: COMPACT SYMPLECTIC TORIC ORBIFOLDS

**9.1. Pullback and pushforward maps for inclusions of polytopes.** In this section, we collect the notions of pullback and pushforward maps of Stanley-Reisner rings. Let  $\Delta$  be a simple polytope with  $m$  facets  $H_1, \dots, H_m$ . For  $\tau \in K_{\Delta}$ , let  $G = \cap_{i \in \tau} H_i$  be an  $(n-r)$ -dimensional face of  $\Delta$ . Then  $G$  is also a simple polytope and the corresponding simplicial complex  $K_G$  is isomorphic to the link  $K_{\Delta, \tau}$  of  $\tau$  in  $K_{\Delta}$ , namely,  $K_{\Delta, \tau} := \{\sigma \subset [m] \setminus \tau \mid \sigma \sqcup \tau \in K_{\Delta}\}$ . Let  $K_{\Delta, \tau} * \tau$  be the joint of  $K_{\Delta, \tau}$  with the simplex  $\tau$ , namely  $K_{\Delta, \tau} * \tau := \{\sigma_1 \sqcup \sigma_2 \mid \sigma_1 \in K_{\Delta, \tau} \text{ and } \sigma_2 \subset \tau\}$ . Then

$$\widetilde{\text{SR}}(G) := \text{SR}(K_{\Delta, \tau} * \tau) \cong \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle x^{\sigma} \mid \sigma \subset [m] \setminus \tau, \sigma \cup \tau \in K_{\Delta} \rangle} \cong \text{SR}(G) \otimes \mathbb{Z}[x_i, i \in \tau],$$

where  $x^{\sigma} := \prod_{i \in \sigma} x_i$ . If  $G' = \cap_{i \in \tau'} H_i$  be a non-empty face contained in  $G$ , i.e.  $\tau \subset \tau'$ , then naturally  $K_{\Delta, \tau'} * \tau'$  is a subcomplex of  $K_{\Delta, \tau} * \tau$ . Thus there are natural pullback and pushforward maps on the Stanley-Reisner rings:

$$(i_{G', G})^* : \widetilde{\text{SR}}(G) \twoheadrightarrow \widetilde{\text{SR}}(G'), \quad x_i \mapsto x_i, \quad (i_{G', G})_* : \widetilde{\text{SR}}(G') \rightarrow \widetilde{\text{SR}}(G), \quad 1 \mapsto x^{\tau' \setminus \tau}.$$

The pushforward is determined by the image of 1 since the pullback map is a surjective ring homomorphism and the pushforward is a homomorphism as  $\widetilde{\text{SR}}(G)$ -module where the module structure on  $\widetilde{\text{SR}}(G')$  is induced by the pullback map.

**9.2. Equivariant Chen-Ruan orbifold cohomology.** We use the notation from Section 6. Let  $\mu : M \rightarrow \mathfrak{t}^*$  be the moment map for the toric orbifold  $[M/\mathbf{S}]$  so that  $\mu(M) = \Delta$ . Recall from [LT] that  $'B \cdot () + \eta$  embeds  $\Delta$  into  $\mathfrak{t}^*$ , and  $M$  is defined as the preimage of  $\Delta' := 'B(\Delta) + \eta$  under the standard moment map  $\bar{\mu} : \mathbb{C}^m \rightarrow \mathfrak{t}^*$ . The moment map  $\mu$  is the composition of  $\bar{\mu}$  with the inverse of  $'B \cdot () + \eta$  restricted to  $\Delta' := 'B \cdot (\Delta) + \eta$ . The faces of  $\Delta'$  are given by the intersections of  $\Delta'$  and the coordinate planes in  $\mathfrak{t}^* \cong \mathbb{R}^n$ .

**Lemma 9.1.** *Let  $G := H_{j_1} \cap \cdots \cap H_{j_r}$  be an  $(n - r)$ -dimensional face. The global stabilizer  $\mathbb{T}_{\mu^{-1}(G)}$  of  $\mu^{-1}(G)$  in  $\mathbb{T}$  is*

$$\mathbb{T}_{\mu^{-1}(G)} = \{(t_1, \dots, t_m) \in \mathbb{T} \mid t_i = 1, \forall i \in [m] \setminus \{j_1, \dots, j_r\}\} = \bigcap_{v: \text{ vertex of } G} \mathbb{T}_{F_v}.$$

Furthermore  $H_{\mathbb{T}}(\mu^{-1}(G)) \cong \widetilde{\mathbf{SR}}(G)$ .

*Proof.* By definition, we have  $\mu^{-1}(G) = \{z \in \mathbb{C}^m \mid |z_i|^2 = ('B \cdot v + \eta)_i, 1 \leq i \leq m, v \in G\}$ . Thus  $z \in \mu^{-1}(G^\circ)$  if and only if  $z_i = 0$  for  $i \in \{j_1, \dots, j_r\}$ , where  $G^\circ$  is the relative interior of  $G$ . The second claim follows from the similar calculation as in Section 6.  $\square$

Let  $g := (g_1, \dots, g_m) \in \mathbf{S} \subset \mathbb{T}$ . By the lemma above,  $M^g = \mu^{-1}(G_g)$  where  $G_g$  is the union of faces  $G$  such that  $\mathbb{T}_{\mu^{-1}(G)}$  contains  $g$ . Let  $\bar{a}_g := \{i \mid g_i = 1\}$ ,  $\bar{b}_g := \{i \mid g_i \neq 1\} \subset \{1, \dots, m\}$ . Then  $g \in \mathbb{T}_{\mu^{-1}(G)}$  if and only if  $\bar{b}_g \subset \{j_1, \dots, j_r\}$ . Therefore we have

**Lemma 9.2.**  $G_g = \bigcap_{i \in \bar{b}_g} H_i$ . In particular,  $H_{\mathbb{T}}^*(M^g) = \widetilde{\mathbf{SR}}(G_g)$ .

Now the intersection of  $M^g$  and  $M^h$  is given by  $M^g \cap M^h = \bar{\mu}^{-1}(G_g \cap G_h)$  where  $G_g \cap G_h = \bigcap_{i \in \bar{b}_g \cup \bar{b}_h} H_i$ . Thus the normal bundle of  $M^{g,h}$  in  $M$  is given by  $\bigoplus_{i \in \bar{b}_g \cup \bar{b}_h} \mathbb{C} \cdot \frac{\partial}{\partial z_i}$ . Let  $\lambda_i \in \text{Hom}(\mathbb{T}, S^1)$  such that  $\lambda_i(t) = t_i$  and define  $0 \leq \tilde{\lambda}_i(g) < 1$  by  $\lambda_i(g) = e^{2\pi i \tilde{\lambda}_i(g)}$ . Thus the obstruction bundle and the virtual class are

$$\mathcal{R}(g, h) = \bigoplus_{\substack{\tilde{\lambda}_i(g) + \tilde{\lambda}_i(h) + \tilde{\lambda}_i((gh)^{-1}) = 2, \\ i \in \bar{b}_g \cup \bar{b}_h}} \mathbb{C} \cdot \frac{\partial}{\partial z_i}, \quad \text{and} \quad c_{\mathbb{T}}(g, h) = \prod_{\substack{\tilde{\lambda}_i(g) + \tilde{\lambda}_i(h) + \tilde{\lambda}_i((gh)^{-1}) = 2, \\ i \in \bar{b}_g \cup \bar{b}_h}} x_i.$$

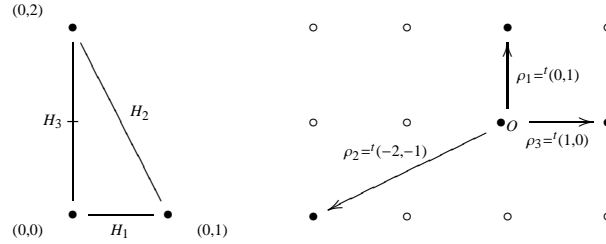
The normal bundle of  $M^{g,h}$  in  $M^{gh}$  is given by  $\bigoplus_{i \in (\bar{b}_g \cup \bar{b}_h) \setminus \bar{b}_{gh}} \mathbb{C} \cdot \frac{\partial}{\partial z_i}$  and its Euler class is  $\prod_{i \in (\bar{b}_g \cup \bar{b}_h) \setminus \bar{b}_{gh}} x_i$ . The equivariant Chen-Ruan cohomology space is  $H_{\mathbf{R}, \text{orb}}([M/\mathbf{S}]) = \bigoplus_{g \in \mathbf{S}} \widetilde{\mathbf{SR}}(G_g)$  where  $\widetilde{\mathbf{SR}}(G_g) := 0$  if  $G_g = \phi$ . Since the pullback and pushforward maps of the equivariant cohomology agree with the ones on the Stanley-Reisner rings, we find that the product is given by

$$1_g \odot 1_h = \underbrace{\left( \prod_{\substack{\tilde{\lambda}_i(g) + \tilde{\lambda}_i(h) + \tilde{\lambda}_i((gh)^{-1}) = 2, \\ i \in \bar{b}_g \cup \bar{b}_h}} x_i \right)}_{\text{virtual class}} \cdot \underbrace{\left( \prod_{i \in (\bar{b}_g \cup \bar{b}_h) \setminus \bar{b}_{gh}} x_i \right)}_{\text{Euler class of normal bundle}} \cdot 1_{gh},$$

where  $1_g, 1_h$  and  $1_{gh}$  are the identities in the corresponding Stanley-Reisner ring. The product  $\cdot$  on the right-hand side can be defined as the product in  $\mathbb{Z}[x_1, \dots, x_m]$ , and each sector is generated by the identity element as a  $\mathbb{Z}[x_1, \dots, x_m]$ -algebra. Thus the above formula is enough to compute the general product.

**9.3. Demonstration of computations.** In this section, we present two computations, namely the weighted projective spaces in dimension 2, with weights  $(1, 1, 2)$  and  $(1, 2, 4)$ .

**9.3.1. The weighted projective space  $\mathbb{P}_{(1,1,2)}^2$ .** Consider the following polytope with facets  $H_i$ , facet labels all (implicitly) 1, and the corresponding primitive inward-pointing normal vectors  $\rho_i$  to the facets.



The polytope is given by

$$\Delta = \{v \in \mathbb{R}^2 \mid \langle \rho_i, v \rangle \geq -\eta_i, i = 1, 2, 3\}$$

where  $(\eta_1, \eta_2, \eta_3) = (0, 2, 0)$ . The corresponding matrix  $B$  is  $\begin{pmatrix} 0 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$  and  $A$  is  $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ .

Thus  $M$  is given by  $|z_1|^2 + |z_2|^2 + 2|z_3|^2 = 2$  in  $\mathbb{C}^3$  and  $\mathbf{S} = \{(t, t, t^2) \mid t \in \mathbb{U}(1)\} \subset \mathbb{T} = \mathbb{U}(1)^3$ . The only elements  $g$  of  $\mathbf{S}$  such that  $G_g$  is not empty are

$$(1, 1, 1) \quad (-1, -1, 1)$$

and the corresponding  $\bar{b}_g, G_g, K_{G_g}, K_{G_g} * \bar{b}_g, \tilde{\lambda}_i$  and ages are given in the following table.

$g$	$\bar{b}_g$	$G_g$	$K_{G_g}$	$K_{G_g} * \bar{b}_g$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	2age
$\mathbf{1} := (1, 1, 1)$	$\phi$	$\Delta$	$K_\Delta$	$K_\Delta$	0	0	0	0
$\sigma := (-1, -1, 1)$	$\{1, 2\}$	$\bullet$	$\phi$	$\bullet\text{---}\bullet$	1/2	1/2	0	2

(9.1)

Thus

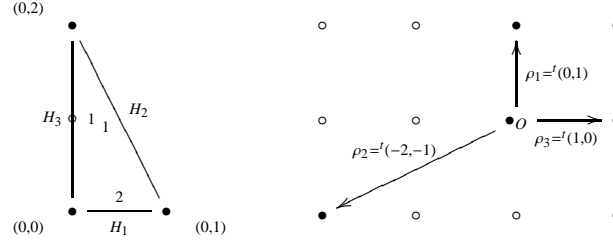
$$H_{CR,R}([M/\mathbf{S}]) = \underbrace{\frac{\mathbb{Z}[x_1, x_2, x_3]}{\langle x_1 x_2 x_3 \rangle}}_{\mathbf{1}} \oplus \underbrace{\frac{\mathbb{Z}[x_1, x_2, x_3]}{\langle x_3 \rangle}}_{\sigma}.$$

The following is the table of the multiplications between  $1_{\mathbf{1}}$  and  $1_{\sigma}$ .

	$1_{\mathbf{1}}$	$1_{\sigma}$
$1_{\mathbf{1}}$	$1_{\mathbf{1}}$	$1_{\sigma}$
$1_{\sigma}$	$1_{\sigma}$	$(1) \cdot (x_1 x_2) \cdot 1_{\mathbf{1}}$

(9.2)

9.3.2. *The weighted projective space  $\mathbb{P}_{(1,2,4)}^2$ .* Consider the following polytope with facets  $(H_1, H_2, H_3)$  and with the labels  $(1, 1, 2)$  respectively, and the corresponding primitive inward-pointing normal vectors  $\rho_i$  of facets.



The corresponding matrix  $B$  is  $\begin{pmatrix} 0 & -2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$  and  $A$  is  $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ . The hyperplanes defining  $\Delta$  is still given by  $(\eta_1, \eta_2, \eta) = (0, 2, 0)$ . Thus  $\mathbf{S} = \{(t, t^2, t^4)\}$  and  $M$  is given by the equation  $|z_1|^2 + 2|z_2|^2 + 4|z_3|^2 = 4$ . The corresponding  $\bar{b}_g, G_g, K_{G_g}, K_{G_g} * \bar{b}_g, \tilde{\lambda}_i$  and ages are computed in the following table.

$g$	$\bar{b}_g$	$G_g$	$K_{G_g}$	$K_{G_g} * \bar{b}_g$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	2age
$\mathbf{1} = (1, 1, 1)$	$\phi$	$\Delta$	$K_\Delta$	$K_\Delta$	0	0	0	0
$\xi = (\sqrt{1}, -1, 1)$	$\{1, 2\}$	$\bullet$	$\phi$	$\bullet\text{---}\bullet$	1/4	1/2	0	3/2
$\xi^2 = (-1, 1, 1)$	$\{1\}$	$\bullet\text{---}\bullet$	$\bullet\bullet$	$\bullet\text{---}\bullet\text{---}\bullet$	1/2	0	0	1
$\xi^3 = (-\sqrt{1}, -1, 1)$	$\{1, 2\}$	$\bullet$	$\phi$	$\bullet\text{---}\bullet$	3/4	1/2	0	5/2

(9.3)

Thus

$$H_{CR,R}([M/\mathbf{S}]) = \underbrace{\frac{\mathbb{Z}[x_1, x_2, x_3]}{\langle x_1 x_2 x_3 \rangle}}_{\mathbf{1}} \oplus \underbrace{\frac{\mathbb{Z}[x_1, x_2, x_3]}{\langle x_3 \rangle}}_{\xi} \oplus \underbrace{\frac{\mathbb{Z}[x_1, x_2, x_3]}{\langle x_2 x_3 \rangle}}_{\xi^2} \oplus \underbrace{\frac{\mathbb{Z}[x_1, x_2, x_3]}{\langle x_3 \rangle}}_{\xi^3}.$$

The following is the table of the multiplications of  $\mathbf{1}, \mathbf{1}_\xi, \mathbf{1}_{\xi^2}, \mathbf{1}_{\xi^3}$ :

$g \backslash h$	$\mathbf{1}$	$\mathbf{1}_\xi$	$\mathbf{1}_{\xi^2}$	$\mathbf{1}_{\xi^3}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}_\xi$	$\mathbf{1}_{\xi^2}$	$\mathbf{1}_{\xi^3}$
$\mathbf{1}_\xi$	$\mathbf{1}_{\xi^2}$	$(1) \cdot (x_2) \cdot \mathbf{1}_{\xi^2}$	$\mathbf{1}_{\xi^3}$	$(1) \cdot (x_1 x_2) \cdot \mathbf{1}$
$\mathbf{1}_{\xi^2}$	$\mathbf{1}_{\xi^2}$	$\mathbf{1}_{\xi^3}$	$(1) \cdot (x_1) \cdot \mathbf{1}$	$(x_1) \cdot (1) \cdot \mathbf{1}_\xi$
$\mathbf{1}_{\xi^3}$	$\mathbf{1}_{\xi^3}$	$(1) \cdot (x_1 x_2) \cdot \mathbf{1}$	$(x_1) \cdot (1) \cdot \mathbf{1}_\xi$	$(x_1) \cdot (x_2) \cdot \mathbf{1}_{\xi^2}$

9.4. **Presentations of  $H_{R,orb}([M/\mathbf{S}])$  as a subring.** From our main result,  $H_{R,orb}^*([M/\mathbf{S}])$  is a subring of

$$\mathcal{NH}_R^*(\nu[F/\mathbf{S}]) = \bigoplus_{g \in \mathbf{S}} \bigoplus_{v \in G_g} \mathbb{Z}[x_i, i \in \mathbf{v}] = \bigoplus_{g \in \mathbf{S}} \left\{ (p_v)_{\bar{b}_g \subset \mathbf{v}}, p_v \in \mathbb{Z}[x_i, i \in \mathbf{v}] \right\}$$

where  $\mathbf{v}$  corresponds to  $v$  by  $v = \bigcap_{i \in \mathbf{v}} H_i$ . Note that  $v \in G_g \Leftrightarrow \bar{b}_g \subset \mathbf{v}$ . The product  $(p_v)_{\bar{b}_g \subset \mathbf{v}} \star (p_w)_{\bar{b}_h \subset \mathbf{w}} \in \bigoplus_{\bar{b}_{gh} \subset \mathbf{u}} \mathbb{Z}[x_i, i \in \mathbf{u}]$  can be computed by its  $u$ -component

$$\begin{aligned} & (p_v)_{\bar{b}_g \subset \mathbf{v}} \star (p_w)_{\bar{b}_h \subset \mathbf{w}} \Big|_u \\ &= p_u \cdot q_u \cdot \left( \prod_{\substack{\bar{\lambda}_i(g) + \bar{\lambda}_i(h) + \bar{\lambda}_i((gh)^{-1}) = 2, \\ i \in \bar{b}_g \cup \bar{b}_h}} x_i \right) \left( \prod_{i \in (\bar{b}_g \cup \bar{b}_h) \setminus \bar{b}_{gh}} x_i \right) \end{aligned}$$

if  $\bar{b}_g \cup \bar{b}_h \subset \mathbf{u}$  and otherwise is zero.

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